Supplemental Notes

Essentials of Business Mathematics

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Contents

1 Review of Arithmetic

1.1 Basics of Arithmetic

1.1.1 Simple order of operations

One of the more fundamentally important aspects of this course is the **Order of Operations**, which you should be familiar with from previous mathematics courses. If the Order of Operations is not understood clearly and memorized, the result will likely be many simple mistakes in calculations. So spend some time and make sure that you can work through the calculations of this and the next several sections correctly using the Order of Operations.

Here we give the order of operations:

1. Perform all operations(according to the order of operations) within parenthesis/brackets.
2. Perform operations with exponents.
3. Perform multiplication or division whichever comes first from left to right.
4. Perform addition and subtraction whichever comes first from left to right.

Consider using "BEDMAS" or "PEMDAS" to help you remember the correct Order of Operations.

Let us look at some examples:

Example 1.1.1. Compute: $(10 - 4) \div 2$

Solution:

$$\begin{aligned}(10 - 4) \div 2 &= 6 \div 2 \\ &= 3\end{aligned}$$

So our answer is 3. ◀

Example 1.1.2. Compute: $20 \div (7 + 3) \times 2 - 3$

Solution:

$$\begin{aligned}20 \div (7 + 3) \times 2 - 3 &= 20 \div 10 \times 2 - 3 \\ &= 2 \times 2 - 3 \\ &= 4 - 3 \\ &= 1\end{aligned}$$

So our answer is 1. ◀

Example 1.1.3. Compute: $48 \div (3 \times 2^3) - 6$

Solution:

$$\begin{aligned} 48 \div (3 \times 2^3) - 6 &= 48 \div (3 \times 8) - 6 \\ &= 48 \div 24 - 6 \\ &= 2 - 6 \\ &= -4 \end{aligned}$$

So our answer is -4. ◀

Example 1.1.4. Compute: $\dfrac{19-4}{4^2-11}$

Solution: Initial inspection of this problem may lead one to think that the order of operations is not clear on how to proceed. When dealing with fractions (which really mean division), we always completely evaluate the numerator and denominator before dividing. In terms of mathematics:

$$\begin{aligned} \frac{19-4}{4^2-11} &= (19-4) \div (4^2 - 11) \\ &= 15 \div (16 - 11) \\ &= 15 \div 5 \\ &= 3 \end{aligned}$$

So our answer is 3. ◀

Our final example should be considered more challenging. Note that sometimes we use parenthesis or brackets to denote multiplication instead because, the symbol "$\times$", looks a lot like "x".

Example 1.1.5. Compute: $(7 - 3(2)) - [6 - (5^2 - 2)]$

Solution:

$$\begin{aligned} (7 - 3(2)) - [6 - (5^2 - 2)] &= (7 - 6) - [6 - (25 - 2)] \\ &= 1 - (6 - 23) \\ &= 1 - (-17) \\ &= 1 + 17 \\ &= 18 \end{aligned}$$

So our answer is 18. ◀

Examples 1.1.4 and 1.1.5 should be studied closely as the ideas in each of these problems will come up repeatedly in this course, other courses, and in the application of business mathematics. Pay special attention to homework problems that look similar to these.

1.2 Fractions

1.2.1 Common fractions

Fractions or **Common Fractions** is a general term used to describe a set of numbers quantitatively indicating parts of a whole. For example, 2/5, means 2 parts out of a whole of 5 and 3/7 means 3 parts out of a whole of 7. The number written above the dividing line is called the **numerator**. The number written below the dividing line is called the **denominator**. The numerator and denominator are also sometimes called the **terms of the fraction**. Notice in the previous sentence we called the horizontal line a "dividing" line. This is because fractions are in fact all division operations. In other words, 2/5 means exactly the same thing as $2 \div 5$ and 3/7 means exactly the same thing as $3 \div 7$.

There are two types of fractions. A **proper fraction** has a numerator that is less than the denominator. For example, 3/7 is a proper fractions. An **improper fraction** has a numerator that is greater than a denominator. 4/3 is an example of an improper fraction.

In conclusion and to be clear, each of these arithmetic expressions have exactly the same meaning.

$$1/3 = \frac{1}{3} = {}^{1}\!/_{3} = 1 \div 3$$

1.2.2 Equivalent fractions

Equivalent fractions may be generated by multiplying some fraction by some form of the number one. There are many useful applications for the generation of equivalent fractions.

Let us motivate how to correctly and logically generate equivalent fractions. First, note that we multiply two fractions by multiplying the numerators to get the resultant numerator and then multiply the denominators to get the resultant denominator. In mathematical terms, let a, b, c, and d be numbers, then:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \tag{1.1}$$

Also note that if we multiply a number (or fraction) with 1, we do not change it:

$$\begin{aligned} \frac{a}{b} &= \frac{a}{b} \times 1 \\ &= \frac{a}{b} \times \frac{1}{1} \\ &= \frac{a \times 1}{b \times 1} && \text{By equation 1.1 above} \\ &= \frac{a}{b} \end{aligned}$$

Notice that we ended with exactly the same thing that we started with. In the examples below we will see how we can take advantage of the above fact in concert with the fact that $1 = \frac{a}{a}$, for all a to make fractions look different but not really be different.

Example 1.2.1. Consider $\dfrac{2}{3}$. Find an equivalent fraction with a numerator of 8.

Solution: Notice in the problem that the numerator is 2. We need to think of a way to change it to 8 using stardard arithmetic that gives us an equivalent fraction. But at the same time to change the 2 to an 8 we would have to multiply 2 times 4. Looking at the previous discussion we know that we can multiply a number(or fraction) times 1, giving us an equivalent fraction. Further, notice that $1 = \frac{4}{4}$. Let us look at the mathematics involved in this:

$$\begin{aligned}\frac{2}{3} &= \frac{2}{3} \times 1 \\ &= \frac{2}{3} \times \frac{4}{4} \\ &= \frac{2 \times 4}{3 \times 4} \\ &= \frac{8}{12}\end{aligned}$$

Thus we have generated an equivalent fraction with numerator 8. Our answer is $\frac{8}{12}$. ◀

Example 1.2.2. Consider $\dfrac{1}{5}$. Find an equivalent fraction with a denominator of 15.

Solution: We need to change the denominator from 5 to 15. Notice that $5 \times 3 = 15$. So multiply by $\frac{3}{3}$.

$$\begin{aligned}\frac{1}{5} &= \frac{1}{5} \times 1 \\ &= \frac{1}{5} \times \frac{3}{3} \\ &= \frac{1 \times 3}{5 \times 3} \\ &= \frac{3}{15}\end{aligned}$$

Thus we have generated an equivalent fraction with denominator 15. Our answer is $\frac{3}{15}$. ◀

The homework problems in the textbook will present these problems in a slightly different way. It will tell you what to multiple the numerator and denominator by to generate the equivalent fraction.

Next, we will look at how equivalent fractions can be used to **simplify** fractions. Fractions certainly can be useful in the form resulting from the examples above (see adding and subtracting fractions by generating a common denominator). However, generally speaking a fraction is said to be simplifiable or reducable if there is a common divisor in the numerator and denominator. The process of reducing or simplifying involves dividing out the common divisor. We will be effectively reversing the steps in examples 1.2.1 and 1.2.2.

Example 1.2.3. Simplify $\frac{12}{32}$.

Solution: Simplifying a fraction may be done in one or more steps depending what you notice are common divisors. For example notice that 2 is a common divisor of the numerator and denominator.

$$\begin{aligned}\frac{12}{32} &= \frac{6\times 2}{16\times 2}\\ &= \frac{6}{16}\times\frac{2}{2}\\ &= \frac{6}{16}\times 1\\ &= \frac{6}{16}\end{aligned}$$

But notice that there is still 2 common. So we have not simplified completely.

$$\begin{aligned}\frac{6}{16} &= \frac{3\times 2}{8\times 2}\\ &= \frac{3}{8}\times\frac{2}{2}\\ &= \frac{3}{8}\times 1\\ &= \frac{3}{8}\end{aligned}$$

We now know that we have completely simplified because the only common divisor between the numerator and denominator is 1. Our answer is $\frac{3}{8}$. ◀

Some of you may have noticed that this problem could have been shorter if we had noticed that 4 was in common from the beginning. Students should take advantage of the largest or greatest common divisor to make your work easier, if you can find it.

Also, some of you may be use to dividing out the common divisor instead of converting to 1. This is mathematically consistent and is certainly ok to use (the textbook uses this method).

Example 1.2.4. Simplify $\frac{210}{252}$

Solution: Notice that 42 is a common multiple.

$$\begin{aligned}\frac{210}{252} &= \frac{5\times 42}{6\times 42}\\ &= \frac{5}{6}\times\frac{42}{42}\\ &= \frac{5}{6}\times 1\\ &= \frac{5}{6}\end{aligned}$$

So our answer is $\frac{5}{6}$. ◀

1.2.3 Converting fractions into decimal form

Common fractions can be converted to decimal equivalents or approximations using the long division process that you would have learned in grade school. However, long division can be somewhat time consuming and a calculator can give us the decimal equivalent or approximation quickly. Thus, students should use calculators to save time and avoid careless errors. None-the-less, we briefly show the long division procedure in an example:

Example 1.2.5. Find the decimal equivalent of $\frac{3}{4}$.

Solution: Use long division to divide 4 into 3:

$$\begin{array}{r}0.75\\ 4\,\overline{)\,3.00}\\ \underline{2.8}\\ 0.20\\ \underline{0.20}\\ 0.00\end{array}$$

Thus the decimal equivalent of $\frac{3}{4}$ is 0.75. ◀

Further examples will make use of a calculator and indicate how we deal with repetition in decimal values.

Example 1.2.6. Find the decimal equivalent of $\frac{19}{20}$.

Solution:

$$\begin{aligned}\frac{19}{20} &= 19\div 20\\ &= 0.95\end{aligned}$$

So our decimal equivalent is 0.95. ◀

Example 1.2.7. Find the decimal equivalent of $\frac{2}{3}$.

Solution:

$$\begin{aligned}\frac{2}{3} &= 2 \div 3 \\ &= 0.\overline{6}\end{aligned}$$

So our answer is $0.\overline{6}$. ◀

Example 1.2.8. Find the decimal equivalent of $\frac{997}{999}$.

Solution:

$$\begin{aligned}\frac{997}{999} &= 997 \div 999 \\ &= 0.\overline{997}\end{aligned}$$

So our answer is $0.\overline{997}$ ◀

Example 1.2.9. Find the decimal equivalent of $\frac{7}{6}$.

Solution:

$$\begin{aligned}\frac{7}{6} &= 7 \div 6 \\ &= 1.1\overline{6}\end{aligned}$$

So our answer is $1.1\overline{6}$ ◀

We conclude this subsection by letting the student know that the textbook will use a slightly different notation for indicating a repeating decimal value. The book will place a small point over each digit that repeats. Either notation will be understood by the instructor for tests and quizzes.

1.2.4 Converting mixed numbers to decimal form

Mixed numbers are numbers that have both whole numbers and fraction parts, such as $3\frac{2}{3}$. 3 is the whole number part and $\frac{2}{3}$ is the fraction part.

Definition 1. *Let a be a whole number and $\frac{b}{c}$ be a fraction, with $c \neq 0$. We call $a\frac{b}{c}$ a Mixed Number and define it according to the equation:*

$$a\tfrac{b}{c} = a + \tfrac{b}{c}$$

The process of converting mixed numbers to decimal form is simple. We convert the fraction part to decimal form with our calculator, then add the whole number part.

Example 1.2.10. Find the decimal equivalent of $1\frac{3}{4}$.

Solution:

$$\begin{aligned} 1\frac{3}{4} &= 1 + \frac{3}{4} && \text{By definition 1} \\ &= 1 + 0.75 && \text{Convert fraction to decimal} \\ &= 1.75 \end{aligned}$$

So our answer is 1.75. ◀

Example 1.2.11. Find the decimal equivalent of $5\frac{3}{8}$.

Solution:

$$\begin{aligned} 5\frac{3}{8} &= 5 + \frac{3}{8} && \text{By definition 1} \\ &= 5 + 0.375 && \text{Convert fraction to decimal} \\ &= 5.375 \end{aligned}$$

So our answer is 5.375. ◀

Example 1.2.12. Find the decimal equivalent of $2\frac{1}{12}$.

Solution:

$$\begin{aligned} 2\frac{1}{12} &= 2 + \frac{1}{12} && \text{By definition 1} \\ &= 2 + 0.08\overline{3} && \text{Convert fraction to decimal} \\ &= 2.08\overline{3} \end{aligned}$$

So our answer is $2.08\overline{3}$ ◀

1.2.5 Rounding

This subsection will give you a quick refresher on rounding values. In business it is very common for us to deal with dollar values which, for many possible reasons, must be rounded.

Consider for a moment the value sometimes seen on a gasoline station's sign for the price of a liter of fuel. Those values are sometimes given to the tenths of a cent precision. For example: $1.259 per liter. Clearly rounding must occur because one cannot physically pay $\frac{9}{10}$ of a cent. Here I outline the method used by your textbook.

Rounding Procedure

1. If the first digit in the group of decimal digits that is to be dropped is the digit 5, 6, 7, 8, or 9, the last digit retained is increased by 1.
2. If the first digit in the group of decimal digits that is to be dropped is the digit 0, 1, 2, 3, or 4, the last digit retained is left unchanged.

Let us look at some examples.

Example 1.2.13. Round the numbers 5.458, 2.734, and 9.997 to two decimal places.

Solution:

$5.458 \to 5.46$	increase 5 to 6 in the second decimal place
$2.734 \to 2.73$	drop the digit 4
$12.997 \to 13.00$	increase 9 in second decimal place to 10 and add accordingly

The final problem in example 1.2.13 may seem a bit confusing. Just look at it as if you are adding $12.99 + 0.01$. If you do the arithmetic you will see that a 1 has to be carried twice giving us 13.00 or 13.

$$\begin{array}{rr} & 12.99 \\ + & 0.01 \\ \hline & 13.00 \end{array}$$

Our answers are 5.46, 2.73, and 13.00 respectively. ◀

Working problems from the textbook homework will benefit students well as rounding is a common source of errors early in this course.

1.2.6 Complex Fractions

Complex Fractions are algebraic expressions with fractions were the numerator or denominator contain other fractions. Even though such expressions are rather complicated in appearance, they are of considerable importance within finance, as they are used almost universally in the calculation of interest on investments or loans.

Our interest here is how do we calculate with complex fractions and, more precisely, how do they fit into the order of operations? A good rule to follow is to deal with the nested fraction first and work your way out with respect to the order of operations.

Example 1.2.14. [1, p. 6]Simplify the expression $\frac{1600}{223 \times \frac{3}{8}}$ and round your answer to two decimal places.

Solution:

$$\begin{aligned} \frac{1600}{223 \times \frac{3}{8}} &= \frac{1600}{223 \times 0.375} && \text{Convert the nested fraction to a decimal} \\ &= \frac{1600}{83.625} \\ &= 19.133034\ldots && \text{Record many digits to avoid error in answer} \\ &\approx 19.13 \end{aligned}$$

Our rounded answer is 19.13. ◀

Example 1.2.15. Simplify the expression $675 \times \left(\frac{4 + \frac{1}{8}}{\frac{7}{3} - 1} \right)$ and round your answer to two decimal places.

Solution:

$$
\begin{aligned}
675 \times \left(\frac{4+\frac{1}{8}}{\frac{7}{3}-1}\right) &= 675 \times \left(\frac{4+0.125}{2.\overline{3}-1}\right) && \text{Convert the nested fractions to decimals} \\
&= 675 \times \left(\frac{4.125}{1.\overline{3}}\right) \\
&= 675 \times 3.09375 && \text{Record many digits to avoid error in answer} \\
&= 2088.28125 \\
&\approx 2088.28
\end{aligned}
$$

Our rounded answer is 2088.28. ◀

Example 1.2.16. Simplify the expression $\dfrac{1255}{1-0.24\times\frac{210}{365}}$ and round your answer to two decimal places.

Solution:

$$
\begin{aligned}
\frac{1255}{1-0.24\times\frac{210}{365}} &= \frac{1255}{1-0.24\times 0.\overline{57534246}} && \text{Convert the nested fraction to a decimal} \\
&\approx \frac{1255}{1-0.1380821904} && \text{Record many digits to avoid error in answer} \\
&\approx \frac{1255}{0.8619178096} \\
&\approx 1456.055306\ldots \\
&\approx 1456.06
\end{aligned}
$$

Our rounded answer is 1456.06. ◀

Complex fractions are among the most error prone topics/problems in any mathematics course in secondary and higher education mathematics courses. Generally students should not leave out steps.

As far as long decimal values are concerned the rules for how many decimals to keep depend on the number of precision digits (significant digits) involved in the calculation. For this course you should preserve at least 4 digits following the decimal point up until rounding your final answer. Truncating calculations further could result in penalizable errors in answers. In the business and finance community when reckless truncation of decimals occur the result is potentially significant accounting problems.

1.3 Percent

Percents are used nearly universally within business mathematics, finance, and accounting. Informally the word percent means, out of one hundred. However we now give it formal meaning.

Definition 2. *Let p be some real number such that $p \geq 0$, p* **percent** *or p % is defined according to the equation*

$$p\ \% = \frac{p}{100}$$

This just means that all percents are common fractions with numerator p and denominator 100. In other words, all percents are out of a whole of 100.

1.3.1 Writing percents as fractions

Converting percents to fractions is a natural first step in this section as the process is more or less obvious from the definition above.

Example 1.3.1. Convert 23% to a fraction in simplified form.

Solution: Using definition 2 we proceed.

$$23\% = \frac{23}{100}$$

Our answer is $\frac{23}{100}$. ◀

Example 1.3.2. Convert 15% to a fraction in simplified form.

Solution: Using definition 2 we proceed.

$$\begin{aligned} 15\% &= \frac{15}{100} \\ &= \frac{3}{20} && \text{5 is greatest common multiple} \end{aligned}$$

Our answer in simplified form is $\frac{3}{20}$. ◀

1.3.2 Writing percents as decimals

Example 1.3.3. Convert 47% to decimal form.

Solution: Using definition 2 we proceed.

$$\begin{aligned} 47\% &= \frac{47}{100} \\ &= 0.47 && \text{Use calculator or move decimal place} \end{aligned}$$

Our answer is 0.47. ◀

Example 1.3.4. Convert $\frac{3}{10}$% to decimal form.

Solution:

$$\begin{aligned} \frac{3}{10}\% &= \frac{\frac{3}{10}}{100} \\ &= \frac{3}{10} \div 100 \\ &= 0.003 \end{aligned}$$

Our answer is 0.003. ◀

Be very careful with the previous example. It should be studied carefully, as it is the source of many mistakes on quizzes and tests in this course.

Example 1.3.5. Convert 0.0024% to decimal form.

Solution:

$$\begin{aligned} 0.0024\% &= \frac{0.0024}{100} \\ &= 0.000024 && \text{Use calculator or move decimal place} \end{aligned}$$

The correct answer is 0.000024. ◀

1.3.3 Writing decimals and fractions as percents

Here we are simply reversing the process from the previous two subsections above. As a result, instead of dividing by 100, we now multiply by 100.

Example 1.3.6. Convert 0.69 to percent form.

Solution:

$$\begin{aligned} 0.69 &= 0.69 \times 100\% \\ &= 69\% \end{aligned}$$

Our answer is 69%. ◀

Example 1.3.7. Convert 7.42 to percent form.

Solution:

$$\begin{aligned} 7.42 &= 7.42 \times 100\% \\ &= 742\% \end{aligned}$$

The answer is 742%. ◀

Example 1.3.8. Convert $\frac{8}{9}$ to percent form.

Solution:

$$\begin{aligned} \frac{8}{9} &= \frac{8}{9} \times 100\% \\ &= 88.\overline{8}\% \end{aligned}$$

The correct answer is $88.\overline{8}\%$. ◀

1.4 Applications-Averages and Related Problems

1.4.1 Basic problems

Everyone has worked with averages, especially when trying to figure out our current score in a course in some classes. To compute an average we just take all of the values that we wish to average, add them up, and divide by the total number of values. Here is the precise definition in mathematical language:

Definition 3. *Let x_1, x_2, x_3, … x_n be n numbers. The* **Average** *or mean of these n numbers is*

$$\frac{x_1 + x_2 + x_3 + \ldots x_n}{n}$$

Example 1.4.1. Assume that someone wishes to purchase four watermelons each weighing $3\frac{1}{2}$ kg, $3\frac{3}{8}$ kg, $4\frac{1}{3}$ kg, and 3 kg.

a) What is the average weight of the four watermelons? (round your answer to one decimal place)

b) What is the total value of all of the watermelons, if the price per watermelon is \$1.49?

c) Assuming the price above, what is the average price of the watermelons?

Solution: We work part (a) first. We just need to convert each fraction to a decimal, add everything up, and divide by 4:

$$\begin{aligned} \frac{3\frac{1}{2} + 3\frac{3}{8} + 4\frac{1}{3} + 3}{4} &= \frac{3.5 + 3.375 + 4.\overline{3} + 3}{4} && \text{Convert mixed numbers to decimals} \\ &= \frac{14.208\overline{3}}{4} \\ &= 3.552083\overline{3} \\ &\approx 3.6 \text{ kg} \end{aligned}$$

Next, part (b). We need to find the price if we purchased all of the watermelons. First, we should find the price of each watermelon by multiplying the weight of each watermelon times the price (rounding to the nearest cent):

$$\begin{gathered} 3.5 \times 1.49 = \$5.22 \\ 3.375 \times 1.49 = \$5.03 \\ 4.\overline{3} \times 1.49 = \$6.46 \\ 3 \times 1.49 = \$4.47 \end{gathered}$$

Thus the total value of all four of the watermelons is: $5.22 + 5.03 + 6.46 + 4.47 = \$21.18$ Finally, part (c) we want to average the prices above. Take note that we already have the sum in the numerator of the average worked out in part (b):

$$\begin{aligned}\frac{21.18}{4} &= 5.295 \\ &\approx \$5.30\end{aligned}$$

So the average price of the watermelons is $5.30. ◀

Example 1.4.2. Use the table below to find:

a) The average price of a TV across all stores.

b) The average revenue from TV's across all stores.

Store	TV's Sold	Price per TV($)
A	15	$729
B	8	$1,099
C	20	$899
D	3	$1,199

Table 1.1: TV's Sold at Local Stores

Solution: We begin with part (a). We just need to average the prices in the 3rd column:

$$\begin{aligned}\frac{729 + 1099 + 899 + 1199}{4} &= \frac{3926}{4} \\ &= \$981.50\end{aligned}$$

For part (b) we will need to know how much revenue each store has generated from selling it's TV's. To find the revenue we multiply the price times the number of TV's sold, then divide by 4:

$$\begin{aligned}\frac{15(729) + 8(1099) + 20(899) + 3(1199)}{4} &= \frac{41304}{4} \\ &= \$10326\end{aligned}$$

The numerator of the left side of the equation above represents the sum of the revenues from TV's at each store. ◀

1.5 Applications-Payroll and Conversion between Units

1.5.1 Unit conversion

This section is mainly about **Unit Conversion** and has general importance across all disciplines of applied mathematics. We will extend much further than the textbook during our lecture and within this document and many of the methods discussed will not be in the textbook. After we finish this section you should not only be able to convert between units of time but between any units given the appropriate **conversion equation**.

There are many examples on unit conversion in this document and in the textbook. Time will not allow us to go over each of them in class. However, the student is encouraged to look through all of the examples available.

Unit conversion is a mathematical process by which we take one unit and convert to another unit using a given or know relationship between the two units, this relationship is called a conversion equation. For example, we know that there are 12 months in one year. So 12 months = 1 year is the conversion equation. With this relation we can easily see how much someone gets paid per month given their yearly salary. Another example of a conversion equation would be 1 Canadian Dollar is the same as 0.76 Euros or 1 Canadian Dollar = 0.76 Euro. This equation can help us to convert between Canadian dollars and Euros easily.

The key thing to remember in this process is that the arithmetic/algebra of units obey the same rules as the arithmetic/algebra of numbers. Now for an example:

Before beginning this problem, students are cautioned to take note of the details, as some students will simply work out a calculation that they have memorized. In fact, it is not uncommon for students to ask the question "Couldn't I have gotten the same answer much faster by dividing by 52?" If you are thinking this after the next problem, then it is very likely that you have missed the point of the problem. The more important thing to understand is the reasoning.

Example 1.5.1. An employee with an annual salary of \$42,350 is paid each week. How much is each weekly paycheck?

Solution: Lets think about what we know in relation to the language in the problem. We need to figure out how much each weekly paycheck is and we know the yearly(annual) salary. Thinking carefully we also know a relationship between years and weeks, 1 year = 52 weeks. What can we do with this?

$$\begin{aligned} 1 \text{ year} &= 52 \text{ weeks} \\ \frac{1 \text{ year}}{1 \text{ year}} &= \frac{52 \text{ weeks}}{1 \text{ year}} && \text{Divide both sides by 1 year} \\ 1 &= \frac{52 \text{ weeks}}{1 \text{ year}} && \text{On the left everything cancels out} \qquad (1.2) \end{aligned}$$

The right hand side of the final equation above is called a **Conversion Factor**. We multiple conversion factors with units to convert them in a specific way.

Note that we could also do some arithmetic on the intial conversion equation to create a different conversion factor:

$$\begin{aligned}
1 \text{ year} &= 52 \text{ weeks} \\
52 \text{ weeks} &= 1 \text{ year} && \text{Exchanges sides of equation} \\
\frac{52 \text{ weeks}}{52 \text{ weeks}} &= \frac{1 \text{ year}}{52 \text{ weeks}} && \text{Divide both sides by 52 weeks} \\
1 &= \frac{1 \text{ year}}{52 \text{ weeks}} && \text{On the left everything cancels out} \qquad (1.3)
\end{aligned}$$

Again, on the right hand side of the bottom equation is another conversion factor. In fact, we can create two conversion factors for every conversion equation.

The final important issue to realize is that we are not really changing anything by converting units. Remember above that we multiple by a conversion factor. If you look at the two conversion factors above (1.2 and 1.3), you will see that they are both equal the number 1 and we already know that multiplying a number times one does not change it. We are just making things appear to be different for the sake of communication. Now let us work the problem.

Let us start with what we are given:

$$\begin{aligned}
\$42350 \text{ annual salary} &= \frac{42350 \text{ dollars}}{\text{year}} \\
&= \frac{42350 \text{ dollars}}{\text{year}} \times 1 \\
&= \frac{42350 \text{ dollars}}{1 \text{ year}} \times \frac{1 \text{ year}}{52 \text{ weeks}} && \text{Equation 1.3} \\
&= \frac{42350 \text{ dollars} \times 1 \text{ year}}{1 \text{ year} \times 52 \text{ weeks}} && \text{Equation 1.1 on page 5} \\
&= \frac{42350 \text{ dollars}}{52 \text{ weeks}} && \text{The year units cancel out} \\
&\approx \frac{814.23 \text{ dollars}}{\text{week}} && \text{Divide the numbers} \\
&\approx \$814.23 \text{ weekly pay}
\end{aligned}$$

So our answer is \$814.23 weekly pay. ◀

The important thing to think about in the conversion process is to figure out what units you want to generate and what units you want to cancel or eliminate. This is why we chose the conversion factor in Equation 1.3. We wanted to eliminate years and end up with weeks. So to eliminate the year units in the denominator we would need year units in the numerator via some multiplication. Notice in Equation 1.3 that the year units are in the numerator as needed.

Example 1.5.2. Assad is paid every 2 weeks. Each paycheck, before taxes are removed, is \$2057.25.

a) What is his annual salary?

b) What is his monthly salary?

Solution: In this problem we are trying to convert from units of two weeks to units of 1 year. Students should understand that there may be many ways to get to the solution. However, there is one conversion that students cannot use. The conversion equation 4 weeks = 1 month is false. It is easy to see why. If we assume that every week has 7 days, then $4 \times 7 = 28$ days. Many months have more than 28 days. It is critical that students remember this as it is the result of many false conversions in this section.

Before we begin the conversion let us take note of some related conversion equations:

$$1 \text{ pay period} = 2 \text{ weeks} \tag{1.4}$$
$$52 \text{ weeks} = 1 \text{ year} \tag{1.5}$$

Using these two conversion equations we will now work part (a):

$$\begin{aligned}
\$2057.25 \text{ per pay peroid} &= \frac{2057.25 \text{ dollars}}{\text{pay period}} \\
&= \left(\frac{2057.25 \text{ dollars}}{\text{pay period}}\right)\left(\frac{1 \text{ pay period}}{2 \text{ weeks}}\right)\left(\frac{52 \text{ weeks}}{1 \text{ year}}\right) \\
&= \frac{(2057.25)(1)(52) \text{ dollars}}{(2)(1) \text{ year}} \\
&= \frac{53488.5 \text{ dollars}}{\text{year}} \\
&= \$53,488.50 \text{ annual salary}
\end{aligned}$$

Notice above that the right two fractions in parenthesis are conversion factors generated by eqauations 1.4 and 1.5. Further, notice that the pay period and weeks units cancel out leaving dollars per year as desired.

Someone might have noticed an alternative conversion equation that could have been used to simplify this: 26 pay periods = 1 year.

Now for part (b) we will use our answer from part (a) above and note that:

$$1 \text{ year} = 12 \text{ months} \tag{1.6}$$

So we must generate a conversion factor that cancels year units and generates month units:

$$\begin{aligned}
\frac{53488.5 \text{ dollars}}{\text{year}} &= \left(\frac{53488.5 \text{ dollars}}{\text{year}}\right)\left(\frac{1 \text{ year}}{12 \text{ months}}\right) && \text{From equation 1.6} \\
&= \frac{(53488.5)(1) \text{ dollars}}{12 \text{ months}} \\
&= \frac{4457.375 \text{ dollars}}{\text{month}} \\
&\approx \$4457.38 \text{ monthly salary}
\end{aligned}$$

So our answer is \$4457.38 monthly salary and concludes example 1.5.2. ◀

Again, students are reminded, the procedure here is far more important to understand than just getting the answer. Create conversion factors that cancel units that you do not want and move toward the units that you need from the problem.

Example 1.5.3. Jean is paid a semi-monthly salary of \$929.00 and works a regular workweek of 40 hours. What is her hourly rate of pay?

Solution: We will need several conversion equations here:

$$\begin{aligned} 2 \text{ pay periods} &= 1 \text{ month} \\ 1 \text{ year} &= 12 \text{ months} \\ 1 \text{ year} &= 52 \text{ weeks} \\ 1 \text{ week} &= 40 \text{ work hours} \end{aligned}$$

So,

$$\begin{aligned} \$929.00 \text{ per pay period} &= \frac{929 \text{ dollars}}{\text{pay period}} \\ &= \left(\frac{929 \text{ dollars}}{\text{pay period}}\right)\left(\frac{2 \text{ pay periods}}{1 \text{ month}}\right)\left(\frac{12 \text{ months}}{1 \text{ year}}\right)\left(\frac{1 \text{ year}}{52 \text{ weeks}}\right)\left(\frac{1 \text{ week}}{40 \text{ work hours}}\right) \\ &= \frac{(929)(2)(12)(1)(1) \text{ dollars}}{(1)(1)(52)(40) \text{ work hours}} \\ &= \frac{22296 \text{ dollars}}{2080 \text{ work hours}} \\ &\approx \frac{10.71923 \text{ dollars}}{\text{work hour}} \\ &\approx \$10.72 \text{ per hour} \end{aligned}$$

Thus our answer is \$10.72 per hour. ◀

Hopefully this problem begins to convince you that this process is very nessessary in general.

The next problem is a somewhat different but a related application of the unit conversion procedure.

Example 1.5.4. You wish to travel to Buffalo, NY, US and intend to make purchases of no more than \$300 USD. How much money should you take with you in Canadian Dollars if the conversion rate is \$1.00 CAD is the same as \$1.035 USD?

Solution: Our conversion equation is given at the end of the problem:

$$1.00 \text{ CAD} = 1.035 \text{ USD}$$

Therefore,

$$\begin{aligned}
\$300.00 \text{ USD} &= \frac{300 \text{ USD}}{1} \times \frac{1.00 \text{ CAD}}{1.035 \text{ USD}} \\
&= \frac{(300)(1.00) \text{ CAD}}{(1)(1.035)} \\
&= \frac{300 \text{ CAD}}{1.035} \\
&\approx \$289.85507\ldots \text{ CAD} \\
&\approx \$289.86 CAD
\end{aligned}$$

Our answer is \$289.86. ◀

1.5.2 Commission problems

Example 1.5.5. Jim is paid commission on his sales at company X. He is paid 2.5% on the first \$5000 of sales, 4% on the next \$3000, and 7% on any sales more than \$10,000. How much commission should he be paid in October, if he sold \$12,000 of produce for company X?

Solution: Here we are looking at a graduated commission(see textbook page 21). Basically, Jim is paid a higher commission for more sales.

There are three components to his commission because his sales were in excess of \$10,000. The first \$5000 has a commission of $0.025(5000) = 125$. The next component is 4% on the next \$3000, so $0.04(3000) = 120$. Finally, any amount <u>over</u> \$10,000 he gets commission of 7%, so $0.07(2000) = 140$.

We just need to add these up: $125 + 120 + 140 = \$385$. ◀

1.6 Applications-Taxes

There are many different types of taxes that could be seen in Canada. Students should use the textbook as reference on any problem that does not explicitly indicate the type and percentage of a tax. On quizzes and tests students can expect to see this information given within the problem or on a formula sheet attached.

1.6.1 Sales and service taxes

Example 1.6.1. You make purchases totaling \$362.25 before taxes. How much sales tax would you pay if these purchases where made in:

a) Ontario(13% HST)?

b) Saskatchewan(5% PST)?

c) Quebec(5% GST, then 8.5% PST)?

Solution: Let us begin with part (a). Ontario has a 13% HST (Harmonized Sales Tax). Thus to calculate the taxes we multiply the price times the HST percentage: Therefore,

$$\begin{aligned}\$362.25 \times 13\% &= 362.25 \times 0.13 \\ &= 47.0925 \\ &= \$47.09\end{aligned}$$

If we wanted to know the total amount paid(taxes and price), we would just add the original amount to the taxes.

Now for part (b). In Saskatchewan consumers pay 5% PST (Provincial Sales Tax). We calculate this the same way we calculated the Ontario HST:

$$\begin{aligned}\$362.25 \times 5\% &= 362.25 \times 0.05 \\ &= 18.1125 \\ &= \$18.11 \qquad (1.7)\end{aligned}$$

Finally part (c). Quebec has a more complicated method for calculating its sales tax. Initially GST(5%) is calculated afterwhich consumers must pay PST(8.5%) on the purchase <u>and on the amount of GST.</u> So we must begin by calculating the GST on \$362.25. We would multiply this amount with 5%, which we have already done in equation (1.7) above. This gives us \$18.11 in GST. Next, we calculate the total PST.

$$\begin{aligned}\$362.25(8.5\%) + \$18.11(8.5\%) &= 362.25(.085) + 18.11(.085) \\ &= 32.3306 \\ &= \$32.33 \text{ in PST}\end{aligned}$$

The total amount of sales tax in Quebec would be $18.11+32.33 = \$50.44$. It is important to read the problem carefully to understand the ordering and method of tax calculation. This concludes problem 1.5.1. ◀

1.6.2 Property tax

Property taxes are generally easy to calculate in Ontario. We use the following formula:

$$\text{Property Tax} = \text{Property Value} \times \frac{\text{Mill Rate}}{1000} \qquad (1.8)$$

Be aware that not all municipalities use this formula.

Example 1.6.2. What will Juan pay in property tax in Ontario, if the local mill rate is 11.75 and his property is valued at \$485,000? What will he pay in property tax if the local mill rate is dropped to 10.05?

Solution: Using formula (1.8) we proceed.

$$\begin{aligned}\text{Property Tax} &= 485000 \times \frac{11.75}{1000} \\ &= 5698.75 \\ &= \$5698.75\end{aligned}$$

So the answer to the first question was \$5698.75.

If the mill rate is changed to 10.05 we have

$$\begin{aligned}\text{Property Tax} &= 485000 \times \frac{10.05}{1000} \\ &= 4874.25 \\ &= \$4874.25\end{aligned}$$

So the answer to the second question is \$4874.25. ◀

2 Review of Basic Algebra

Many students come into this course with the assumption that there is a "math part" and a "business part" and that if we can get through the algebra/math part, then the rest is easy. Unfortunately this assumption is incorrect and is somewhat comparable to believing that one can launch a satellite into orbit without computer systems, which would depend on a great deal of luck.

It is very important that students understand that the mathematics and the applications in business are completely inseparable. To make careful decisions in all fields of business we need a precise langauge for communicating our ideas. There are several parts of the language of mathematics and a significant and fundamental portion of this language is called Algebra.

2.1 Algebraic Expressions and Polynomials

Algebraic Expressions come in a huge variety. We deal with a small subset of these expressions in this course. We will call this subset of expressions **Polynomials**. Here we present many examples of polynomials:

a) $6x + 0.77y - 2$

b) $\frac{2}{3}a^3b^2 - 4ab + a^2c$

c) $x^2 - 5x + 2$

d) $6y^3 + 4y^2 - 3y - 7$

e) $5x^2y - 3x + 2x^2y - 4xy^2 + 3$

The critical issue with Polynomials is that we have a combination of variables and/or numbers (not nessessarily both) where the variables have exponents that are whole numbers. As a reminder, we have listed the set of Whole Numbers.

$$\text{Whole Numbers} = \{0, 1, 2, 3 \ldots\}$$

Polynomials have discrete components: terms, coefficients, and constants. **Terms** are the pieces that are added and/or subtracted. **Coefficients** are the numbers multiplied with the variables. **Constants** are terms that contain no variables. In example (a) above the terms are $6x$, $0.77y$, and 2; the coeffcients are 6 and 0.77; the constant is 2. It is also possible to interprete the final term (the constant) as -2 instead of 2. Including the sign is optional but your interpretation should be consistent. Thus if you wish to interpret everything as addition then some of your terms will be negative.

Before we talk about addition and subtraction let us make a definition.

Definition 4. *A set of terms are called* **Like Terms** *if and only if each term has the same variable and exponent combination.*

Referring to example (e) in the examples on the previous page note that the first and third terms have the same variable and exponent combinations: $5x^2y$ and $2x^2y$. This makes them like terms. Note that the fourth term has the same variables but different exponents on the x and y respectively: $-4xy^2$. Since there is no other term with this variable and exponent combination, we say this term does not have a like term. Further, note that we effectively ignore coefficients when checking for like terms.

2.1.1 Addition and subtraction

Like terms are important in algebra because we can add and subtract like terms by simply adding or subtracting the coefficients and keeping the same variable and exponent combination.

Example 2.1.1. Add $7m + 2n + m + 5n$

Solution: Using definition 3 we proceed.

$$\begin{aligned} 7m + 2n + m + 5n &= 7m + m + 2n + 5n && \text{Collect like terms} \\ &= (7+1)m + (2+5)n && \text{Add coefficients of like terms} \\ &= 8m + 7n \end{aligned}$$

So our answer is $8m + 7n$. ◀

Example 2.1.2. Add: $(8x^2 + x - 4) + (-3x^2 + 4x - 2)$

Solution:

$$\begin{aligned} (8x^2 + x - 4) + (-3x^2 + 4x - 2) &= 8x^2 - 3x^2 + x + 4x - 4 - 2 && \text{Collect like terms} \\ &= (8-3)x^2 + (1+4)x - 6 && \text{Add coefficients of like terms} \\ &= 5x^2 + 5x - 6 \end{aligned}$$

So our answer is $5x^2 + 5x - 6$. ◀

Subtraction of polynomials must be handled more carefully. When a $(-)$ sign preceeds a bracketed/parenthetical polynomial expression we must distribute the minus sign as a multiple of -1 to each term within brackets or parenthesis. See the next couple of examples for details.

Example 2.1.3. Subtract: $(8t^2 + 3t - 4) - (3t^2 - 7t + 1)$

Solution:

$$\begin{aligned} (8t^2 + 3t - 4) - (3t^2 - 7t + 1) &= 8t^2 + 3t - 4 - 3t^2 + 7t - 1 && \text{Distribute multiple of } -1 \\ &= 8t^2 - 3t^2 + 3t + 7t - 4 - 1 && \text{Collect like terms} \\ &= 5t^2 + 10t - 5 \end{aligned}$$

So our answer is $5t^2 + 10t - 5$. ◀

Example 2.1.4. Subtract: $-3a - (4a + 2b - c) + (a + 3b + 2c) - (4a - 3c)$

Solution:

$$\begin{aligned} -3a - (4a + 2b - c) + (a + 3b + 2c) - (4a - 3c) &= -3a - 4a - 2b + c + a + 3b + 2c - 4a + 3c \\ &= (-3 - 4 + 1 - 4)a + (-2 + 3)b + (1 + 2 + 3)c \\ &= -10a + b + 6c \end{aligned}$$

And our answer is $-10a + b + 6c$. ◀

2.1.2 Multiplication

Multiplication of polynomials is simple for polynomials with a single term (Monomials) but is more complicated for products of polynomials in general.

Multiplication of Monomials

For multiplication of monomials we simply multiply the numerical coefficients and the the variables.

Example 2.1.5. Multiply: $3x(5y)$

Solution:

$$\begin{aligned} 3x(5y) &= (3 \times 5)(xy) \\ &= 15xy \end{aligned}$$

So our answer is $15xy$. ◀

Example 2.1.6. Multiply: $(-2)(5x)(-3xy)$

Solution:

$$\begin{aligned} (-2)(5x)(-3xy) &= [-2 \times 5 \times (-3)](xxy) \\ &= 30x^2y \end{aligned}$$

So our answer is $30x^2y$. ◀

Multiplication of Monomials with Polynomials

In this case we are dealing with a situation similar to the distribution that took place with subtraction of polynomials, in fact the processes are related. When multiplying a monomial with a polynomial we must distribute the monomial, via multiplication, to each term inside the polynomial.

Example 2.1.7. Multiply: $2a(3a^2 - 4a + 2)$

Solution:

$$\begin{aligned} 2a(3a^2 - 4a + 2) &= 2a(3a^2) + 2a(-4a) + 2a(2) \\ &= 6a^3 - 4a^2 + 4a \end{aligned}$$

So our answer is $6a^3 - 4a^2 + 4a$. ◀

Example 2.1.8. Multiply: $-3x^2y(4xy^2 + x^2y^2 - xy + 1)$

Solution:

$$\begin{aligned} -3x^2y(4xy^2 + x^2y^2 - xy + 1) &= -3x^2y(4xy^2) - 3x^2y(x^2y^2) - 3x^2y(-xy) - 3x^2y(1) \\ &= -12x^3y^3 - 3x^4y^3 + 3x^3y^2 - 3x^2y \end{aligned}$$

Our answer is $-12x^3y^3 - 3x^4y^3 + 3x^3y^2 - 3x^2y$. ◀

Multiplication of a Polynomial with a Polynomial

This will be the most complicated case in this section. To understand how to multiply in the case where we have a polynomial times a polynomial, we need one of the fundamental properties from algebra.

If a, b, and c are real numbers, then

$$a(b + c) = ab + ac \tag{2.1}$$

This property is sometimes called the **Distributive Property**. To help us understand how this property is useful consider the somewhat simple case of a binomial times a binomial:

Example 2.1.9. Multiply: $(4x + 3)(2x + 5)$

Solution: Note that we have not encountered something like this before (unless you have seen it in a previous algebra course). Let us try to rewrite it with a substitution. Let $a = 4x + 3$, $b = 2x$, and $c = 5$. Thus we have

$$\begin{aligned} (4x + 3)(2x + 5) &= a(b + c) && \text{From the substitution above} \\ &= ab + ac && \text{Property 2.1 above} \\ &= (4x + 3)(2x) + (4x + 3)(5) && \text{Remove the substitution} \\ &= 8x^2 + 6x + 20x + 15 && \text{Multiply} \\ &= 8x^2 + 26x + 15 && \text{Collect and combine like terms} \end{aligned}$$

So our answer is $8x^2 + 26x + 15$. ◀

Some of you may recognize that we have derivied a procedure called **FOIL**. F means First, O means outer, I means inner, and L means last. This is a useful shortcut for multiplying binomials. The above procedure can be generalized easily to allow for multiplication of any pair of polynomials.medskip
Now for some examples.

Example 2.1.10. Multiply $(3k - j)(k + 2j)$ using FOIL.

Solution:

$$\begin{aligned}(3k - j)(k + 2j) &= (3k)k + (3k)(2j) + (-j)(k) + (-j)(2j) \\ &= 3k^2 + 6jk - jk - 2j^2 \\ &= 3k^2 + 5jk - 2j^2 && \text{Combine like terms}\end{aligned}$$

So or answer is $3k^2 + 5jk - 2j^2$. ◀

2.1.3 Division

We will not be expanding on division of general polynomials in this course, as the pricess involved has little application in business. We will only look at examples of monomials divided by monomials.

Example 2.1.11. Divide $(25xy) \div (5y)$

Solution:

$$\begin{aligned}(25xy) \div (5y) &= \frac{25xy}{5y} \\ &= \frac{25}{5} \times \frac{xy}{y} \\ &= 5x\end{aligned}$$

So our answer is $5x$. ◀

For the next example we will need a property for addition and subtraction of fractions.

Let a, b, c be real numbers, and $c \neq 0$; then

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \qquad \text{and} \tag{2.2}$$

$$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c} \tag{2.3}$$

Example 2.1.12. Divide $(16n + 8) \div (4)$

Solution:

$$\begin{aligned}(16n + 8) \div (4) &= \frac{16n + 8}{4} \\ &= \frac{16n}{4} + \frac{8}{4} && \text{By property 2.2 above} \\ &= 4n + 2\end{aligned}$$

So our answer is $4n + 2$. ◀

Example 2.1.13. Divide $(8k^3 - 20k^2 + 32k) \div (2k)$

Solution:

$$\begin{aligned}(8k^3 - 20k^2 + 32k) \div (2k) &= \frac{8k^3 - 20k^2 + 32k}{2k} \\ &= \frac{8k^3}{2k} - \frac{20k^2}{2k} + \frac{32k}{2k} \qquad \text{By properties 2.2 and 2.3 above} \\ &= 4k^2 - 10k + 16\end{aligned}$$

So our answer is $4k^2 - 10k + 16$. ◀

2.1.4 Substitution and Evaluation

This aspect of the section should be review from a previous course. It is highly recommended that students who have had little experience or previous trouble in mathematics courses should place all numerical values within brackets or parenthesis to avoid mistakes. See example 2.1.14 below.

Example 2.1.14. Evaluate $4a + 3b - c$ for $a = -1$, $b = 2$, and $c = -3$

Solution:

$$\begin{aligned}4a + 3b - c &= 4(-1) + 3(2) - (-3) \\ &= -4 + 6 + 3 \\ &= 5\end{aligned}$$

So our answer is 5. ◀

Example 2.1.15. Evaluate $P\left(1 + \frac{r}{n}\right)^{nt}$ for $P = 1000$, $r = 0.04$, $n = 4$, and $t = 10$. You will need a calculator.

Solution: We need to proceed carefully and remember the order of operations.

$$\begin{aligned}P\left(1 + \frac{r}{n}\right)^{nt} &= 1000\left(1 + \frac{0.04}{4}\right)^{4\times 10} \\ &= 1000(1 + 0.01)^{40} \\ &= 1000(1.01)^{40} \\ &= 1000(1.488863\ldots) \\ &\approx 1488.86\end{aligned}$$

So our answer is 1488.86. ◀

2.2 Integer Exponents

This section is about algebraic expressions whose variables may have many exponents. Let us begin with the fundamental definition of this section.

Definition 5. *Let a be a real number and n be an element of the set $\{\ldots -2, -1, 0, 1, 2 \ldots\}$ so that a and n are not both 0, then*

$$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors of } a},$$

and call this **a raised to the power of n** *or standard exponential notation.*

The reason we call this section integer exponents is because the set of numbers in this definition (the numbers allowed for exponents) is called the set of **Integers**.

There are several properties of exponents that are helpful in this course. Assume the same about a and n in these properties as the definition above. We will not prove these in class.

1) $a^n a^m = a^{n+m}$	4) $\dfrac{a^n}{a^m} = a^{n-m}$	7) $(a^n)^m = a^{nm}$
2) $(ab)^n = a^n b^n$	5) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	8) $a^0 = 1$
3) $a^{-n} = \dfrac{1}{a^n}$	6) $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	

Table 2.1: Properties of Exponents

In our first two examples we will need to pay close attention to the order of operations.

Example 2.2.1. Simplify, do not leave negative exponents -2^4

Solution: Here notice that the exponent 4 applies only to the 2 immediately to its left. Thus

$$\begin{aligned} -2^4 &= -(2 \times 2 \times 2 \times 2) \\ &= -16 \end{aligned}$$

Thus our answer is -16. ◀

Example 2.2.2. Simplify, do not leave negative exponents $(-2)^4$

Solution: Here the exponent 4 applies to everything inside the parenthesis.

$$\begin{aligned} (-2)^4 &= (-2)(-2)(-2)(-2) \\ &= 16 \end{aligned}$$

Here our answer is 16. ◀

Notice that a small change in symbol combination creates a completely different answer. Make sure you understand the difference between the two problems above, ask questions if nessessary.

Example 2.2.3. Simplify, do not leave negative exponents $4^7 \div 4^4$

Solution:

$$\begin{aligned} 4^7 \div 4^4 &= \frac{4^7}{4^4} \\ &= 4^{7-4} && \text{Property (4) from Table 2.1} \\ &= 4^3 \\ &= 64 \end{aligned}$$

Our answer is 64. ◀

Example 2.2.4. Simplify, do not leave negative exponents $\frac{x^3x^2}{x^4}$

Solution:

$$\begin{aligned} \frac{x^3x^2}{x^4} &= \frac{x^5}{x^4} && \text{Property (1)} \\ &= x^1 && \text{Property (4)} \\ &= x \end{aligned}$$

So our answer is x. ◀

Example 2.2.5. Simplify, do not leave negative exponents $\left(\frac{a^3b^2}{x}\right)^{-2}$

Solution:

$$\begin{aligned} \left(\frac{a^3b^2}{x}\right)^{-2} &= \left(\frac{x}{a^3b^2}\right)^2 && \text{Property (6)} \\ &= \frac{x^2}{(a^3b^2)^2} && \text{Property (4)} \\ &= \frac{x^2}{(a^3)^2(b^2))^2} && \text{Property (2)} \\ &= \frac{x^2}{a^6b^4} && \text{Property (7)} \end{aligned}$$

So our answer is $\frac{x^2}{a^6b^4}$. ◀

Note that there may be many alternative methods for getting to the simplified answer. Also, students do not always need to add every step of detail that the instructor uses in class or within these notes. However, research has shown that the fewer details that are used the more likely a student is to make a mistake.

2.3 Solving Linear Equations in One Variable

Please note that this corresponds to section 2.5 in the textbook. As you may have seen from the schedule we do not cover every section of every chapter.

Our goal in this section is to deal with solving the most simple type of equations. It is important for students to realize that, for the purposes of later material, solving equations is more than just finding the number that balances the equation. Even though this is correct it will help us to think of solving equations in a different light. We will think of solving equations as getting the interested variable by itself (isolated on one side of the equal sign). The question is: how do we get the variable we are interested in by itself? We answer that with some creativity and a set of properties below.Let a, b, and c be real numbers and $a = b$, then

1. $a + c = b + c$
2. $a - c = b - c$
3. $ac = bc$
4. $a \div c = b \div c, \quad$ provided $c \neq 0$

This property is best summarized by saying: We can do anything we want to an equation as long as we do the same thing to both sides.

Example 2.3.1. Solve the equation $x + 5 = 7$.

Solution:

$$\begin{aligned} x + 5 &= 7 \\ x + 5 - 5 &= 7 - 5 && \text{Property 2, subtract 5 on both sides} \\ x &= 2 \end{aligned}$$

So our answer is 2. ◀

Example 2.3.2. Solve the equation $-3y = 12$.

Solution:

$$\begin{aligned} -3y &= 12 \\ \frac{-3y}{-3} &= \frac{12}{-3} && \text{Property 4, divide } -3 \text{ on both sides} \\ y &= -4 \end{aligned}$$

Our answer is -4. ◀

Example 2.3.3. Solve the equation $-x = -5x + 16$.

Solution:

$$\begin{aligned}
-x &= -5x + 16 \\
-x + 5x &= -5x + 5x + 16 && \text{Property 1, add } 5x \text{ on both sides} \\
4x &= 16 \\
\frac{4x}{4} &= \frac{16}{4} && \text{Property 4, divide 4 on both sides} \\
x &= 4
\end{aligned}$$

Here our answer is 4. ◀

Example 2.3.4. Solve the equation $\frac{1}{5}a = 2$.

Solution:

$$\begin{aligned}
\frac{1}{5}a &= 2 \\
5\frac{1}{5}a &= 5(2) && \text{Property 3, multiply 5 on both sides} \\
a &= 10
\end{aligned}$$

Thus we have 10 as our answer. ◀

Example 2.3.5. Solve the equation $3z + 2 = 5z - 5$.

Solution:

$$\begin{aligned}
3z + 2 &= 5z - 5 \\
3z - 5z + 2 &= 5z - 5z - 5 && \text{Property 2, subtract } 5z \text{ on both sides} \\
-2z + 2 &= -5 \\
-2z + 2 - 2 &= -5 - 2 && \text{Property 2, subtract 2 on both sides} \\
-2z &= -7 \\
\frac{-2z}{-2} &= \frac{-7}{-2} && \text{Property 4, divide } -2 \text{ on both sides} \\
z &= \frac{7}{2}
\end{aligned}$$

So we have $\frac{7}{2}$ as our answer. ◀

Example 2.3.6. Solve the equation $x - 12 = 0.6x$.

Solution:

$$\begin{aligned}
x - 12 &= 0.6x \\
x - 0.6x - 12 &= 0.6x - 0.6x && \text{Property 2, subtract } 0.6x \text{ on both sides} \\
0.4x - 12 &= 0 \\
0.4x - 12 + 12 &= 0 + 12 && \text{Property 1, add 12 on both sides} \\
0.4x &= 12 \\
\frac{0.4x}{0.4} &= \frac{12}{0.4} && \text{Property 4, divide 0.4 on both sides} \\
x &= 30
\end{aligned}$$

So our answer is 30. ◀

Example 2.3.7. Solve the equation $3x + 9 - 7x = 24 - x - 3$.

Solution:

$$\begin{aligned}
3x + 9 - 7x &= 24 - x - 3 \\
-4x + 9 &= 21 - x && \text{Combine like terms} \\
-4x + x + 9 &= 21 - x + x && \text{Property 1, add } x \text{ on both sides} \\
-3x + 9 &= 21 \\
-3x + 9 - 9 &= 21 - 9 && \text{Property 2, subtract 9 on both sides} \\
-3x &= 12 \\
\frac{-3x}{-3} &= \frac{12}{-3} && \text{Property 4, divide } -3 \text{ on both sides} \\
x &= -4
\end{aligned}$$

So our answer is -4. ◀

In each of the examples above you should check your solutions by substituting the values we found back into the orginal equation. Students will not be required to check solutions but it is highly recommended.

2.4 Solving More Complicated Equations

Please note that this corresponds to section 2.6 in the textbook and is the final section in chapter 2 that we will cover.

The next step up in difficulty in solving linear equations involves muliplying and/or dividing polynomials. We may also encounter situations where fractions must be delt with. Conveniently, these fractions are nearly always avoidable, when solving equations.

Also note that as we move further into the material more steps will be omitted as they should gradually become obvious, if you are keeping up and doing homework.

Example 2.4.1. Solve the equation $3(x - 4) = -2(3 - 2x)$.

Solution:

$$\begin{aligned}
3(x-4) &= -2(3-2x) \\
3x-12 &= -6+4x && \text{Multiply out the left and right side} \\
3x-3x-12 &= -6+4x-3x && \text{Subtract } 3x \text{ from both sides} \\
-12 &= -6+x \\
-12+6 &= -6+6+x && \text{Add 6 to both sides} \\
-6 &= x
\end{aligned}$$

So our solution is -6. ◀

Example 2.4.2. Solve the equation $x + \frac{5}{8}x = \frac{1}{4}$.

Solution: When trying to solve equations that contain fractions it is always possible to eliminate all fraction by multiplying both sides by the Least Common Denominator or LCD(see textbook). This is also referred to as the least common multiple of the denominators. Some students have trouble finding the LCD. If this is your situation, consider that any common multiple will work, but may require that you simplify your answer at the end. In this problem the LCD happens to be 8.

$$\begin{aligned}
x+\frac{5}{8}x &= \frac{1}{4} \\
8\left[x+\frac{5}{8}x\right] &= 8\left[\frac{1}{4}\right] && \text{Multiply both sides by 8} \\
8x+8\left(\frac{5}{8}x\right) &= 2 && \text{Multiply by distributing on left side} \\
8x+5x &= 2 \\
13x &= 2 \\
\frac{13x}{13} &= \frac{2}{13} && \text{Divide both sides by 13} \\
x &= \frac{2}{13}
\end{aligned}$$

So our solution si $\frac{2}{13}$. ◀

2.4.1 Literal Equations and Formulae

Often times in business and other applications equations come up that have more than one variable. We cannot solve them numerically so we solve them **literallly**. Solving an equation literally, for some variable, means to isolate the variable. Effectively, we are using the same process as above. Note that it is important to remember which variable you are supposed to solve for from the problem.

Example 2.4.3. Solve the equation $S = rt$, for r.

Solution: Since we wish to solve for r, we need to treat the r above like x in the previous problems.

$$\begin{aligned} S &= rt \\ \frac{S}{r} &= \frac{rt}{r} && \text{Divide both sides by } r \\ \frac{S}{r} &= t \end{aligned}$$

So our solution is $t = \frac{S}{r}$. ◀

Example 2.4.4. Solve the equation $A = P(1 + rt)$, for t.

Solution: There are two ways to find t. Each method will give, what look like different answers, but in fact are the same.

$$\begin{aligned} A &= P(1 + rt) \\ A &= P + Prt && \text{Distribute } P \text{ on right side} \\ A - P &= P - P + Prt && \text{Subtract } P \text{ on both sides (need to isolate term containing } t) \\ A - P &= Prt \\ \frac{A - P}{Pr} &= \frac{Prt}{Pr} && \text{Divide both sides by } Pr \\ \frac{A - P}{Pr} &= t \end{aligned}$$

Our solution is $t = \frac{A - P}{Pr}$. ◀

3 Ratios, Proportion, and Percent

Each of the above show up universally across all disciplines. For example, on the stock market values changes of stocks are expressed as a percent increase or decrease.

3.1 Ratios

Ratios are used when comparing relative values of numbers or quantities. As you will see, ratios can be treated, for the most part, as fractions and as a result the rules here will have a similar flavor.

Definition 6. *Let a and b be real numbers, then*

$$a : b = \frac{a}{b}$$

and call it the **ratio** *of a to b, where a and b are called the* **terms** *of the ratio.*

3.1.1 Setting up ratios

Example 3.1.1. At some university the average number of students in a classroom with one teacher is 19. Set this up as a ratio of students to teachers.

Solution: Since it asks for the ratio of student to teachers, we need to put the number of students first (before the colon) and the number of teachers second (after the colon). So the answer is $19 : 1$. ◀

Example 3.1.2. In the production of 1000 widgets a factory manager must spend \$1500 on ingredient A and \$510 on ingredient B. What is the ratio of money spent on ingredient A to money spent on ingredient B for 1000 widgets?

Solution: Here we are looking for the ratio of money spent on ingredient A to ingredient B, thus ingredient A's amount comes first, then ingredient B's amount. So $1500 : 510$ is our answer. ◀

Example 3.1.3. In your pocket you have 2 quarters, 3 dimes, and 1 nichel. What is the ratio relating the quantities of quarters to dimes to nichels? What is the ratio of the value in quarters to the value in dimes to the value in nichels?

Solution: First, we deal with the ratio of quantities. We follow the order in the question. So 2:3:1 is the ratio relating the quantities.

Next, we deal with the ratio of values of each coin. Again we follow the order in the question, but we must take into account the value of each coin. Here we choose to use units of cents. However, one could use units of dollars, but would have to use the methods in the next part of the section to get the same answer. So

$$2(25) : 3(10) : 1(5) = 50 : 30 : 5$$

is the simplified ratio relating the values. ◀

3.1.2 Reducing ratios

Considering that definition 6 equates ratios of two terms to common fractions, then simplifying a ratio is performed in the same way as we simplify a fraction.

Example 3.1.4. Reduce each ratio:

a) 20:15

b) 18:52

c) 17:40

d) 9:27:81

Solution: One does not have to convert each othe above ratios to fractions, we just need to find a common divisor (preferably the greatest common divisor) of the terms. We will work only example (a) by converting to fractions.

a)

$$\begin{aligned} 20 : 15 &= \frac{20}{15} \\ &= \frac{4}{3} && \text{Greatest common divisor is 5} \\ &= 4 : 3 \end{aligned}$$

We will no longer convert to fractions, as it is not nessessary.

b)

$$18 : 52 = 9 : 26 \qquad \text{Greatest common divisor is 2}$$

c)

$$17 : 40 \qquad \text{Only common divisor is 1, so this does not reduce.}$$

d) In this final example, we will need to find a common divisor for all three terms.

$$9 : 27 : 81 = 1 : 3 : 9 \qquad \text{Greatest common divisor for all terms is 3}$$

This concludes our problem. ◀

3.1.3 Using equivalent ratios in higher terms

As when dealing with fractions, it is sometimes useful to be able to generate ratios in higher terms. The process is the same as with fractions because ratios can be treated, for the most part, like fractions.

Example 3.1.5. State the following ratios without decimals in lowest terms:

a) 6.2:12.4

b) 1.5:6

c) 4.25:3.75

d) 1.8:3.6:4

Solution: The strategy here will be to create higher terms to get rid of decimal values and then reduce to lowest terms.

a)

$$\begin{aligned} 6.2 : 12.4 &= 62 : 124 && \text{Multiply by 10 to eliminate decimals} \\ &= 1 : 2 && \text{Greatest common divisor is 62} \end{aligned}$$

b)

$$\begin{aligned} 1.5 : 6 &= 15 : 60 && \text{Multiply terms by 10} \\ &= 1 : 4 && \text{Greatest common divisor is 15} \end{aligned}$$

c)

$$\begin{aligned} 4.25 : 3.75 &= 425 : 375 && \text{Multiply terms by 100} \\ &= 17 : 15 && \text{Greatest common divisor is 25.} \end{aligned}$$

d)

$$\begin{aligned} 1.8 : 3.6 : 4 &= 18 : 36 : 40 && \text{Multiply all terms by 10} \\ &= 9 : 18 : 20 && \text{Greatest common divisor for all terms is 2} \end{aligned}$$

Concluding our problem. ◀

3.1.4 Applications-allocation problems

An **allocation problem** is an application of ratios where we wish to divide up something according to a ratio.

Steps for Solving Allocation Problems

1. Add the terms of the ratio.
2. Create a fraction for each part to be allocated by taking each term of the original ratio and dividing by the sum of the terms from part (1).
3. Multiply the value to be allocated by the constructed fraction in part (2). The resulting values solve the allocation problem.

Example 3.1.6. Allocate \$675 in the ratio 2:3. Round any decimal values to the nearest cent.

Solution: We follow the steps above.

1. We begin by adding up the terms of the ratio: $2 + 3 = 5$.
2. Construct fractions for each part to be allocated, where the numertor is each term of the orginal ratio and denominator is the sum in part (1): $\frac{2}{5}$ and $\frac{3}{5}$.
3. We wish to allocate \$675, so we will multiply this by each fraction in part (2):

$$\frac{2}{5} \times 675 = 270 = \$270.00$$

$$\frac{3}{5} \times 675 = 405 = \$405.00$$

We have allocated \$675 in the ratio 2 : 3. Note that if you simplify the ratio 270 : 405 you will get 2 : 3. This is a good way to check your answer, if you have time. ◀

Example 3.1.7. Isaac has \$125,000 that he wishes to invest in mutual funds, ETF's, and money market accounts in the ratio 4 : 3 : 5 respectively. How much should be invested in each?

Solution: To be clear the word "respectively" here means that the terms of the ratio reflect the ordering of the langauage for each investment type.

1. We begin by adding up the terms of the ratio: $4 + 3 + 5 = 12$.
2. Construct fractions for each part to be allocated, where the numertor is each term of the orginal ratio and denominator is the sum in part (1): $\frac{4}{12}$, $\frac{3}{12}$, and $\frac{5}{12}$.
3. We wish to allocate \$125,000, so we will multiply this by each fraction in part (2):

$$\frac{4}{12} \times 125000 = 41666.\overline{6} = \$41,666.67 \qquad \text{to mutual funds}$$
$$\frac{3}{12} \times 125000 = 31250 = \$31,250 \qquad \text{to ETF's}$$
$$\frac{5}{12} \times 125000 = 52083.\overline{3} = \$52,083.33 \qquad \text{to money markets}$$

We can check the answer by adding up the amounts. We should get about $125,000. ◀

The next example tends to cause problems for students. Please realize that it should not be treated any different than the last problems.

Example 3.1.8. Jane wishes to set up a living will so that her descendants: Anna, Lucy, and Steve receive the money from her bank accounts in the ratio $\frac{1}{2} : \frac{2}{3} : \frac{1}{3}$ respectively. If Jane were to pass away today, how much would each descendant receive? Assume the total value in Jane's bank accounts is $20,360?

Solution: In problems like this, it will be extremely important to record all decimal places until our final answer. This problem will not be too bad but many of your homework, quiz, and test problems could raise issues.

1. We begin by adding up the terms of the ratio: $\frac{1}{2} + \frac{2}{3} + \frac{1}{3} = 1.5$.
2. Construct fractions for each part to be allocated, where the numertor is each term of the orginal ratio and denominator is the sum in part (1): $\frac{1/2}{1.5}$, $\frac{2/3}{1.5}$, and $\frac{1/3}{1.5}$.
3. We wish to allocate $20,360, so we will multiply this by each fraction in part (2):

$$\frac{1/2}{1.5} \times 20360 = 6786.\overline{6} = \$6786.67 \qquad \text{to Anna}$$
$$\frac{2/3}{1.5} \times 20360 = 9048.\overline{8} = \$9048.89 \qquad \text{to Lucy}$$
$$\frac{1/3}{1.5} \times 20360 = 4524.\overline{4} = \$4524.44 \qquad \text{to Steve}$$

This concludes our problem. ◀

3.2 Proportions

Definition 7. *Any equation resulting from the equality of ratios is called a* **Proportion**.

3.2.1 Solving proportions

Here we motivate a fast method for solving proportions.

Proposition 1. *Let a, b, c, and d be real numbers such that b and d are not zero, then $a : b = c : d$ is the same thing as $ad = bc$.*

Proof.

$$\begin{aligned} a : b &= c : d \\ \frac{a}{b} &= \frac{c}{d} && \text{Definition 6} \\ bd\frac{a}{b} &= bd\frac{c}{d} && \text{Multiply both sides by } bd \\ ad &= bc && \text{Cancel } b \text{ on left and } d \text{ on right and reorder} \end{aligned}$$

Giving us the desired result. □

So, what we can conclude is that a proportion is really the equality of the cross product of the equivalent fraction. This might seem somewhat anti-climatic, however, let us see if it will help in an example.

Example 3.2.1. Solve the proportion $2 : 7 = 5 : x$.

Solution:

$$\begin{aligned} 2 : 7 &= 5 : x \\ 2x &= 7(5) && \text{Proposition 1} \\ 2x &= 35 \\ x &= \frac{35}{2} \\ x &= 17.5 \end{aligned}$$

Thus our answer is 17.5. ◀

Example 3.2.2. Solve the proportion $\frac{1}{3} : \frac{3}{4} = x : \frac{3}{5}$.

Solution:

$$\begin{aligned} \frac{1}{3} : \frac{3}{4} &= x : \frac{3}{5} \\ \frac{1}{3} \times \frac{3}{5} &= \frac{3}{4}x && \text{Proposition 1} \\ \frac{3}{15} &= \frac{3x}{4} \\ 3(4) &= 15(3x) && \text{Cross multiple and set equal} \\ 12 &= 45x \\ \frac{12}{45} &= x \\ 0.2\overline{6} &= x \end{aligned}$$

So our solution is $\frac{12}{45}$ or $0.2\overline{6}$. ◀

Example 3.2.3. Assume that 7 liters of gasoline costs \$8.61, what is the cost of 12 liters?

Solution: We can set this up as a proportion, there are multiple configurations that will work, but be consistent.

$$\begin{aligned} 7 : 8.61 &= 12 : x \\ 7x &= 8.61(12) \\ 7x &= 103.32 \\ x &= \frac{103.32}{7} \\ x &= \$14.76 \end{aligned}$$

So our solution is \$14.76. ◀

Example 3.2.4. Many years of data indicate that at a certain company must spend \$350 in manufacturing products to make 25 widgets. Estimate the manufacturing costs to make 135 widgets.

Solution:

$$\begin{aligned} 350 : 25 &= x : 135 \\ 350(135) &= 25x \\ 47250 &= 25x \\ \frac{47250}{25} &= x \\ \$1890 &= x \end{aligned}$$

So our solution is \$1890. ◀

3.3 The Basic Percentage Problem

To find percentages, we multiply a number by a percent.

50% of $40 = 0.50 \times 40 = 20$

Here 50% is called the rate; 40 is called the base or orginal number; and 20 is called the percentage or new number. So we generalize this in a formula:

$$\text{Percentage} = \text{Rate} \times \text{Base} \tag{3.1}$$

or

$$\text{New Number} = \text{Rate} \times \text{Original Number}$$

Example 3.3.1. Find the percentage or new number for each of the following. Round your answers to two decimal places.

a) 23% of 600

b) 4% of 38

c) 0.3% of \$1252

d) $\frac{2}{5}$% of \$10,240

Solution: In each case we use Formula 3.1.

a)

$$\begin{aligned} 23\% \text{ of } 600 &\implies 0.23 \times 600 \\ &= 138 \end{aligned}$$

b)

$$\begin{aligned} 4\% \text{ of } 38 &\implies 0.04 \times 38 \\ &= 1.52 \end{aligned}$$

c)

$$\begin{aligned} 0.3\% \text{ of } \$1252 &\implies 0.003 \times 1252 \\ &= 3.756 \\ &= \$3.76 \end{aligned}$$

d)

$$\begin{aligned} \frac{2}{5}\% \text{ of } \$10240 &\implies 0.4\% \times 10240 \\ &= 0.004 \times 10240 \\ &= \$40.96 \end{aligned}$$

Conluding our problem. ◀

Example 3.3.2. What percent of 15 is 6?

Solution: We proceed by looking at key words within the problem. The word ”of” means, multiply, and the word ”is” means, equals. ”What percent” implies a variable. Thus

$$\begin{aligned} 15x &= 6 \\ x &= \frac{6}{15} \\ x &= 0.4 \\ x &= 40\% \end{aligned}$$

So our answer is 40%. ◀

Example 3.3.3. 68 is what percent of 95? Round your answer to the nearest percent.

Solution:

$$\begin{aligned} 68 &= 95x \\ \frac{68}{95} &= x \\ 0.715789\ldots &= x \\ 72\% &= x \end{aligned}$$

So our answer is 72%. ◀

Example 3.3.4. What number is 32% of 47?

Solution:

$$\begin{aligned} x &= 0.32 \times 47 \\ x &= 15.04 \end{aligned}$$

So our answer is 15.04. ◀

Example 3.3.5. Two years ago sales at Company W was \$3.85 million. What were last years sales if sales were 5% higher than two years ago?

Solution: We first need to figure out what the sales increase was by multiplying the percent increase times the sales figures from two years ago. Remember that the units are in millions of dollars.

$$0.05(3.85) = \$0.1925 \text{ million}$$

We now need to add this back to the sales from two years ago. Giving us the solution.

$$3.85 + 0.1925 = \$4.0425 \text{ million}$$

We will further generalize this type of problem in the next section. ◀

3.4 Percent increase/decrease

3.4.1 Basic increase/decrease problems

Here we generalize the process for percent increase or percent decrease.

Let x be the "original number", r be the rate(as a decimal) of increase or decrease, and n be the "new number". The formula for computing a percent increase is

$$x + rx = n \tag{3.2}$$

and a percent decrease is

$$x - rx = n \tag{3.3}$$

The rate of change is computed with

$$\text{Rate of Change} = \frac{\text{Amount of Change}}{x} \tag{3.4}$$

Example 3.4.1. 123 increased by 20% is what number?

Solution: 123 is the original number and 0.20 is the rate of change as a decimal. So using equation 3.2 above

$$123 + 0.20(123) = 147.6$$

So our answer is 147.6. ◀

Example 3.4.2. What number is 12% less than 198?

Solution: 198 is the original number and 0.12 is the rate of a change as a decimal. So using equation 3.3 above

$$198 - 0.12(198) = 174.24$$

So our answer is 174.24. ◀

Example 3.4.3. \$678 is what percent more than \$245? Round your answer to the nearest percent.

Solution: Here we use equation 3.4 above. The original amount here is \$245. Note that the amount of change is $678 - 245 = 433$

$$\begin{aligned} \text{Rate of Change} &= \frac{433}{245} \\ &= 1.767347\ldots \\ &\approx 177\% \end{aligned}$$

Our answer is 177% ◀

3.4.2 Finding the original amount

Finding the original amount from an application problem is much more complex than the previous problem set, thus we devote an entire subsection to this. We will have to draw upon our algebra from Chapter 2.

Example 3.4.4. 235 is 25% more than what number?

Solution: We use equation 3.2 from the previous page and note that 235 is the new value and 0.25 is rate.

$$\begin{aligned} x + 0.25x &= 235 \\ 1.25x &= 235 && \text{Add like terms} \\ x &= \frac{235}{1.25} \\ x &= 188 \end{aligned}$$

So our answer is 188. ◀

Example 3.4.5. What amount of money when diminished by 65% is \$100? Round your answer to the nearest cent.

Solution: Here 100 is the new number and 0.65 is the rate of a change as a decimal. So using equation 3.3

$$\begin{aligned} x - 0.65x &= 100 \\ 0.35x &= 100 \\ x &= \frac{100}{0.35} \\ x &= 285.714285\ldots \\ x &= 285.71 \end{aligned}$$

Giving us the solution of 285.71. ◀

3.5 Applications-Percent

In this section we will look at applications of percents and refer to formulae throughout the chapter.

Example 3.5.1. You go into a store and find an item that you wish to purchase. On the item you see a price tag of \$49.99, however, on the rack where the item was found you see a discount sticker which says that all items here are 20% off (meaning you will pay \$49.99 less 20%). Ignoring taxes, how much will you pay for the item?

Solution: Using equation 3.3 from the previous section, \$49.99 is the orginal value, and 0.20 is the rate (of discount).

$$\begin{aligned}49.99 - 0.20(49.99) &= 39.992 \\ &= \$39.99\end{aligned}$$

Solving our problem. ◀

Example 3.5.2. Ned is about to pay for his family's meal in a restaurant. The total bill is \$129.42. He does not know how to calculate a reasonable tip and decides to "wing it". He pays for the bill with credit and leaves \$23.50 on the table for the tip. What percent tip did he leave, approximately?

Solution: Here we use equation 3.4 in the previous section and assume \$129.42 is the original number and the amount of change will be \$23.50. We could also use equation 3.2.

$$\begin{aligned}\text{Rate of Change} &= \frac{23.50}{129.42} \\ &= 0.181579\ldots \\ &= 18\%\end{aligned}$$

Our answer is 18%. ◀

Example 3.5.3. A few years ago you decided to invest some savings into the TSX Composite Index Fund. After the markets had closed later that day you logged onto your accounts and saw that your share of the TSX Fund were valued at \$18,456.25 and noticed that the TSX Composite Index had risen 1.25% from the previous day. How much was your account worth yesterday?

Solution: Here we use equation 3.2 and let \$18,456.25 be the new number and 0.0125 be the rate as a decimal. We are looking for the original number x.

$$\begin{aligned}x + 0.0125x &= 18456.25 \\ 1.0125x &= 18456.25 && \text{Combine like terms} \\ x &= \frac{18456.25}{1.0125} \\ x &= 18228.39506\ldots \\ x &= \$18,228.40\end{aligned}$$

Giving us the solution of \$18228.40. ◀

3.6 Applications-More on Currency Conversion

Students should look back at section 1.5 on Unit conversion when not using a table. We will also commit a bit of time to understanding a currency conversion table. First, we repeat examples that were used early in the course as a reminder of the conversion process, in general.

3.6.1 Conversion without a table

Example 3.6.1. You wish to travel to Buffalo, NY, US and intend to make purchases of no more than \$300 USD. How much money should you take with you in Canadian Dollars if the conversion rate is \$1.00 CAD is the same as \$1.035 USD?

Solution: Our conversion equation is given at the end of the problem:

$$1.00 \text{ CAD} = 1.035 \text{ USD}$$

Therefore,

$$\begin{aligned}
\$300.00 \text{ USD} &= \frac{300 \text{ USD}}{1} \times \frac{1.00 \text{ CAD}}{1.035 \text{ USD}} \\
&= \frac{(300)(1.00) \text{ CAD}}{(1)(1.035)} \\
&= \frac{300 \text{ CAD}}{1.035} \\
&\approx \$289.85507\ldots \text{ CAD} \\
&\approx \$289.86
\end{aligned}$$

So our answer is \$289.86. ◀

3.6.2 Conversion with a table

Here we learn to use **Currency Cross Rates Tables** to convert between currencies. Initially, one needs to be careful about how dated a Cross Rates Table is. Updated tables can be found all over the internet. You will each be given one for exam and quiz problems (if such a problem shows up).

The most important thing to remember about using these tables is the operation used is universally multiplication. One should never divide as we sometimes used in the previous process (see example 3.6.1 above).

	CAD	USD	Euro	GBP	Yen
CAD	1.00	0.9898	1.323	1.562	0.012
USD	1.010	1.00	1.381	1.548	0.011
Euro	0.758	0.724	1.00	1.12	0.008
GBP	0.640	0.646	0.893	1.00	0.007
Yen	83.33	90.91	125.1	142.9	1.00

Table 3.1: Currency Cross Rates

Example 3.6.2. Use Table 3.1 to convert:

1. 325€ to British Pounds.
2. $8025 Canadian to Yen.
3. $95 US to Canadian Dollars.

Solution: When using a cross rates table like the one above we begin by finding the the starting currency along the top row of the table. Next, find the currency we wish to convert to along the left column of the table. Use a ruler or something to find where the row and column meet to find the multiplier.

1. Looking at the table, we wish to convert Euro to GBP. The multiplier is 0.893.

$$325 \times 0.893 = 290.23£$$

2. We wish to convert CAD to Yen. The multiplier is 83.33.

$$8025 \times 83.33 = 668,723.25¥$$

3. We wish to convert USD to CAD The multiplier is 0.9898.

$$95 \times 0.9898 = \$94.03 \text{ US}$$

This concludes our problem and the section. ◀

3.7 Applications-Income Taxes

This section is represented as section 3.8 in the course textbook. We are interested in mathematical application of income taxes. The following table we be important, for reference, thoughout this section.

Tax Brackets	Tax Rates
$40,970 or less	15% of taxable income less than or equal to $40,970; plus
$40,970 to $81,941	22% of taxable income greater than $40,970 and less than or equal to $81,941; plus
$81,941 to $127,021	26% of taxable income greater than $81,941 and less than or equal to $127,021; plus
Over $127,021	29% of taxable income greater than $127,021

Table 3.2: 2010 Federal Income Tax Brackets and Tax Rates

Example 3.7.1. Using table 3.2 above compute federal income taxes due for Anne who had an income in 2010 of $38,250.

Solution: As you will see, income tax computation is very similar to graduated commission computation from chapter 1. The function outlined in the table above is called a **step function** for those of you that have had an advanced functions course in high school. Read line by line through table 3.2, we notice that since her income is less than $40,970, all of it is taxed at 15%. Thus we have a simple problem.

$$38250(.15) = 5737.50$$

So our answer is $5737.50. ◀

Example 3.7.2. How much federal income tax would Alyssa owe in 2010, if her income was $60,000.

Solution: Here we have to proceed carefully as her income will be taxed in two pieces: the amount less than or equal to $40,970 and the amount more. Note that the amount over is $60000 - 40970 = 19030$

$$40970(0.15) + 19030(.22) = \$10332.10$$

So our total tax due is $10332.10. ◀

Example 3.7.3. How much federal income tax would Sahar owe in 2010, if her income was $160,000.

Solution: Here we have to proceed carefully as her income will be taxed in four pieces: the amount less than or equal to $40,970 (which is 40970), the amount between $40,970 and $81,941 ($81941 - 40970 = 40971$), the amount between $81,941 and $127,021

$(127021 - 81941 = 45080)$, and the amount more than \$127,021 $(160000 - 127021 = 32979)$. So her income tax owed will be

$$40970(0.15) + 40971(.22) + 45080(.26) + 32979(.29) = \$36443.83$$

So the total tax due is \$36443.83. ◀

4 Rectangular Coordinates, Linear Equations, and Applications

The order in these notes will be somewhat different from the ordering in the textbook. Students are advised to observe this carefully as to avoid confusion. In particular 4.1 here is the same as 4.2 in the textbook and 4.2 here is the same 4.1 in the textbook, 4.3 is the same as section 6.1. Also keep this in mind when doing the homework.

The ultimate goal in this chapter is to be able to solve problem involving the intersection point of two lines in business applications. However, there will be a bit of build up to understanding that precisely. Students should also be aware that the effective assimilation of 4.1 and 4.2 will help ease us into what is likely the most difficult section in the course (4.3). We begin this assimilation by setting up the standard rectangular coordinate system.

4.1 Graphing Linear Equations

4.1.1 Rectangular coodinates

It is quite arbitrary how we set up our coordinate system, however we will set ours up so that it is consistent with what some of you may have seen in previous mathematics courses. Here we will call the vertical axis the y **axis** and the horizontal axis the x **axis**. The point where the axes cross is called the **origin**.

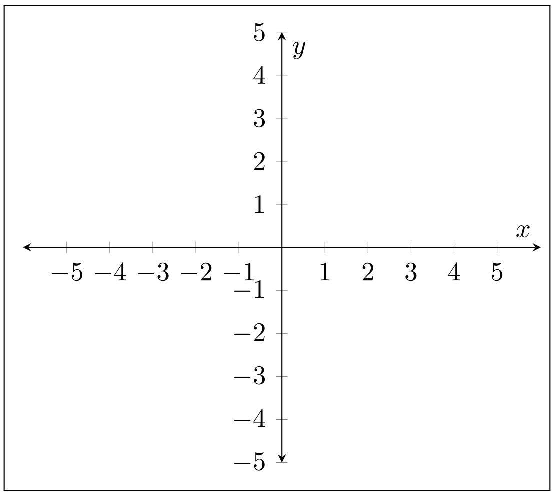
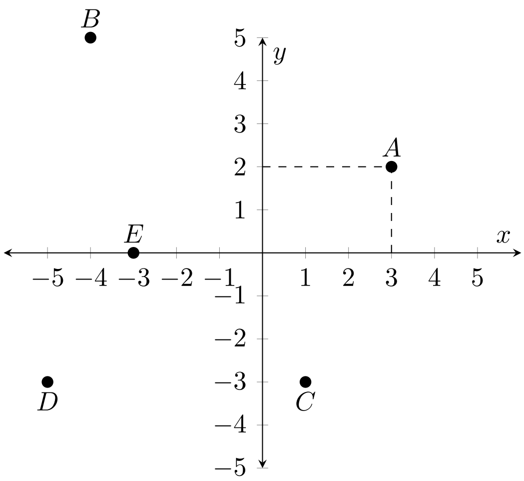


Figure 4.1: Standard Rectangular Coordinate System

We can use the system in figure 4.1 to plot **ordered pairs** for applications later on. Ordered pairs, (x, y), are used to indicate a specific position in this system. The x values tells us how far, left or right, to go on the x- axix and the y value tells us how far to go, up or down, on the y axis. As in the next example, the letter in front of the ordered pair will not always be used, here it clarify which point is which.

Example 4.1.1. Plot the points: $A(3, 2)$; B(-4,5); C(1,-3); D(-5, -3); and E(-3,0) in the rectangular coordinate system.

Solution: Plotting the points we get



Concluding our problem. ◀

4.1.2 Graphing linear equations by making a table of values

Definition 8. *Let x and y be unknown real numbers and a, b, and c be constant real numbers, then any equation that can be written in the form*

$$ax + by = c$$

is called a **linear equation in standard form**.

Our goal here is to use the coordinate system, previously discussed, to generate a way to visualize all of the solutions to equations like the one in definition 8. There are a few methods. We will look at the method that involves constructing a table of x and y values first.

Example 4.1.2. Draw the graph representing all solutions of the equation $-x+y = -1$ by contructing a table.

Solution: The idea here is to find a few solutions to the equation above and interpolate the rest of the solutions by drawing a line in the plane (rectangular coordinate system). To accomplish this we will construct a table of x and y values. We will choose x values near zero, susbstitute them into the equation (or some equivalent form) to get the y values. For this example I will choose -3, -1, 0, 1, 2. In general, as a precaution, choose at least three values.

x	-3	-1	0	1	2
y					

Now to find the y values we can simply substitute the x values in and solve for y, however we will do a bit of algebra to make this easier.

Note that we can solve the equation in the problem for y.

$$\begin{aligned} -x + y &= -1 \\ -x + x + y &= x - 1 && \text{Add } x \text{ to both sides} \\ y &= x - 1 \end{aligned}$$

Now that we have y isolated it will be easier to get them quickly. Begin substituting in the x values, one at a time and do the arithmetic. We will work two examples here, students should confirm the rest of the table independently.

First, letting $x = -3 \implies$

$$\begin{aligned} y &= -3 - 1 \\ &= -4 \end{aligned}$$

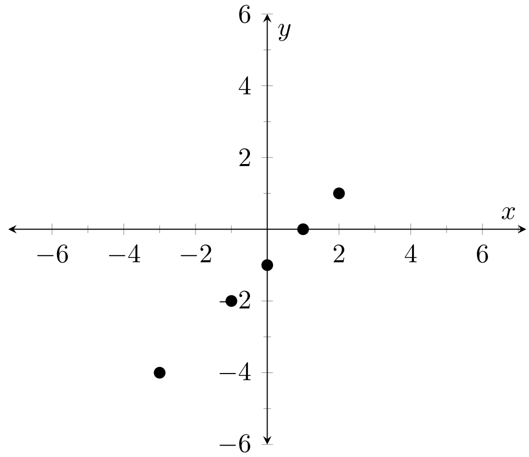
Second, letting $x = -1 \implies$

$$\begin{aligned} y &= -1 - 1 \\ &= -2 \end{aligned}$$

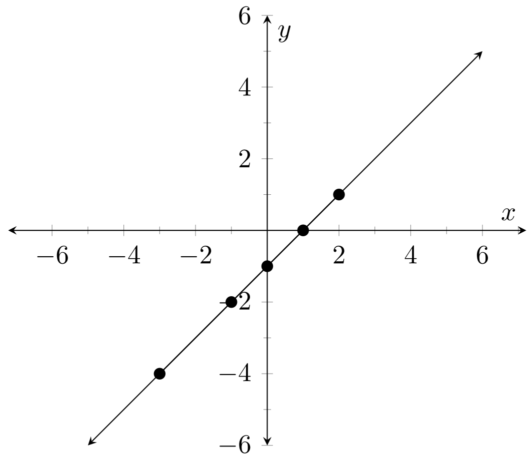
Continuing this process for each choice of x will fill out the above table like this

x	-3	-1	0	1	2
y	-4	-2	-1	0	1

Each column in this table represents an order pair, i.e. $(-5, -6)$ and $(0, -1)$. We should now plot each of these points in the plane.



We can see that these points all line up. The line connecting them from negative infinity to positive infinity is a geometric representation of ALL solutions to the equation $-x + y = -1$. Take close note of the arrows on the ends of the line. Often in application we will not be interested in what happens outside of a certain set of x or y values so the arrows are not critical at this stage.



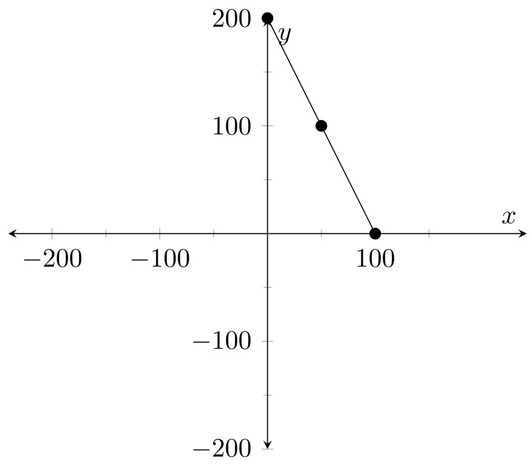
This is our final answer to example 4.1.2. ◀

Example 4.1.3. Graph $y = -2x + 200$ for all values from $x = 0$ to $x = 100$.

Solution: Conveniently in this example we already have the equation solved for y, so we should just choose a few x values. Note that they give us some guidance. We will choose 0, 50, and 100. The table has been constructed and filled out below. Students should verify all values of y from the equation.

x	0	50	100
y	200	100	0

Noting that we are going to have to adjust our scale this time, plotting these ordered pairs, and drawing a connecting line gives us



This is our final graph. Take note that we did not place arrows on the tips of the line because the problem specifically asked for the graph between two points. ◀

4.1.3 Graphing linear equations using slope-intercept form

Many of you have likely seen the methods previously discussed for graphing lines and functions in a previous mathematics course. If this is the case, then you may recall an easier way to graph a line or linear function. We now discuss this alternative method. Be aware that this method will only work with linear equations or lines, which is of course, all we will work with in this course.

To motivate this method we will reference example 4.1.2 from earlier in this section. Looking back at the problem, we decided to solve the equation for y before substituting in the chosen x values.

$$-x + y = -1 \Leftrightarrow y = x - 1$$

Remember that these two forms have the same mathematical meaning. The form on the left was called the standard form of the linear equation. As it turns out, there is a special name for the form on the right (when a linear equation is solved for y).

Definition 9. *Let x and y be unknown real numbers and m and b be constant real numbers, then a linear equation written in the form*

$$y = mx + b$$

is call the **slope-intercept form**. *The number m is called the slope and b the y-intercept.*

As we will soon see graphing a linear equation can be done easily by identifying the slope and y-intercept. The y-intercept or b is simply the location or point where the line crosses the y-axis. The slope or m is a bit more complicated to understand. Consider its definition.

Definition 10. *Let (x_0, y_0) and (x_1, y_1) be two points on a line. The* **slope** *of the line is*

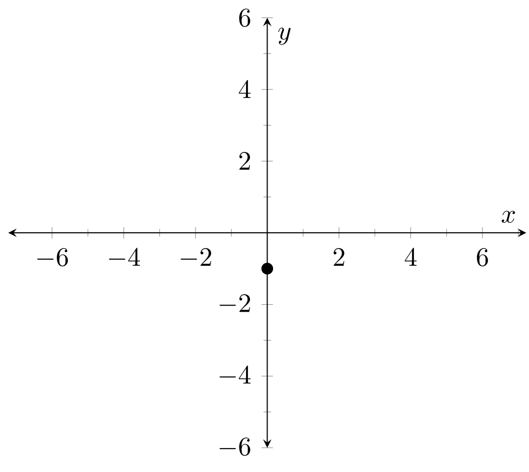
$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\textit{Change in } y}{\textit{Change in } x} = \frac{\textit{Rise}}{\textit{Run}}$$

Any of these interpretations will work fine for this course. Students seem to like the rise over run form. So this is how the in class problems will be motivated.

So going back to example 4.1.2, the right hand side equation is in slope-intercept form. To identify the y-intercept simply read the symbol combination following the x, so the y-intercept is -1. The slope is the number multiplied in front of the x. One might assume that since we do not see a number in front of the x, then there is no slope. However, we should recall that there is an unwritten 1 in front of the x. So the slope is 1. Further, it is always a good idea to interprete the slope as a fraction, referring to definition 10 above. We know that $1 = \frac{1}{1}$, so we have our slope in fraction form. So to summarize:

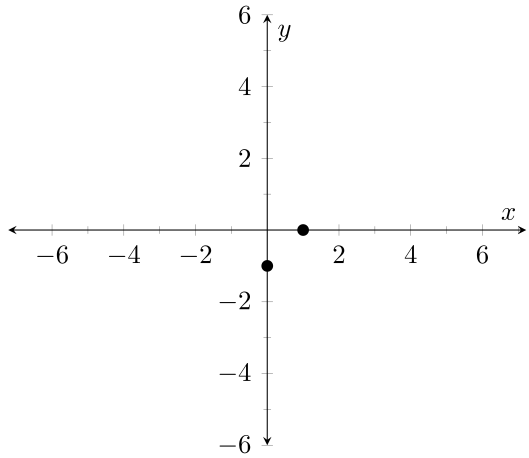
$$y\text{-intercept} = b = -1$$
$$\text{slope} = m = \frac{1}{1}$$

Now let us get to the actual graph of the line using this information. Begin by plotting the y-intercept. Since it is the location where the line crosses the y axis it must be at the coordinates $(0, -1)$.

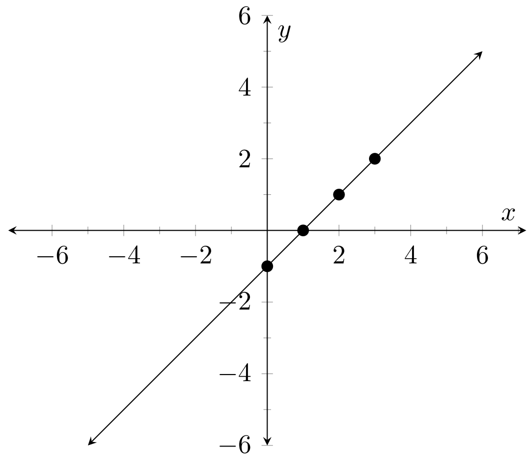


Students should remember that the y-intercept should always be a point on the y axis.

Note that we cannot draw a unique line through this one point, we need another point. We use the slope to get this other point. Starting from the y-intercept, not the origin, Rise 1 and Run 1, because both the numerator and the denominator of the slope is 1. This mean we should plot another point up 1 (because the 1 in the numerator is positive) and right 1 (because the 1 in the numerator is positive). This would give us a new point at $(1, 0)$.



From this new point at $(1, 0)$ we can continue getting more points by moving up 1 and right 1 for as long as we wish. Connecting these points notice that we get the same line that we got in example 4.1.2. Some of the points are different but the line is the same.



This process may seem overwhelming, if you have never seen it before, but in general those that get use to it will likely find that it is far easier and faster than making a table.Let us take a look at a new example.

Example 4.1.4. Graph $3x + y = 4$

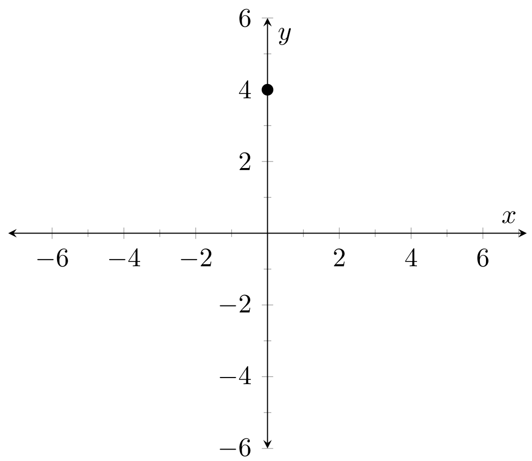
Solution: The first step here will be to write the equation in slope-intercept form by solving for y.

$$\begin{aligned} 3x + y &= 4 \\ y &= -3x + 4 && \text{Subtract } 3x \text{ from both sides} \end{aligned}$$

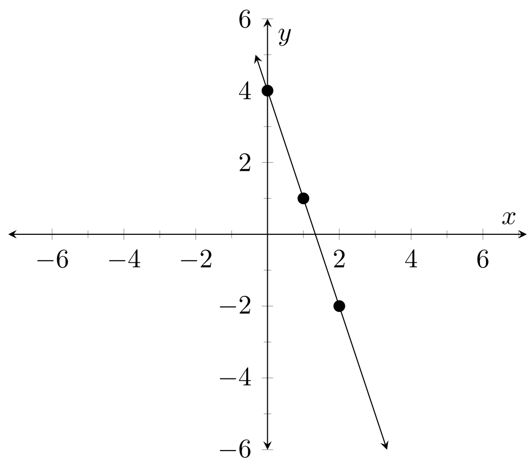
Next, clearly state the slope and y-intercept from the slope-intercept form of the equation from above. Make sure to write the slope as a fraction (use a 1 for the denominator, if it is not a fraction already).

$$\begin{aligned} m &= -3 = \frac{-3}{1} \\ b &= 4 \end{aligned}$$

Thirdly, we start with the y-intercept, remember that this is giving us a point on the y axis, specifically $(0, 4)$



From this point (not the origin) we use the slope to get another point. The -3 in the numerator is telling us to go down 3 (because -3 is a negative number) and right 1 (because 1 is a positive number). This gives us a new point at $(1, 1)$. We plot this point and draw a line to connect them. It is always safe to get at least three points, so we use the slope to get a third point.



This concludes the example. ◀

Example 4.1.5. Graph $x - 3y + 900 = 0$ for all values of x from $x = 0$ to $x = 900$.

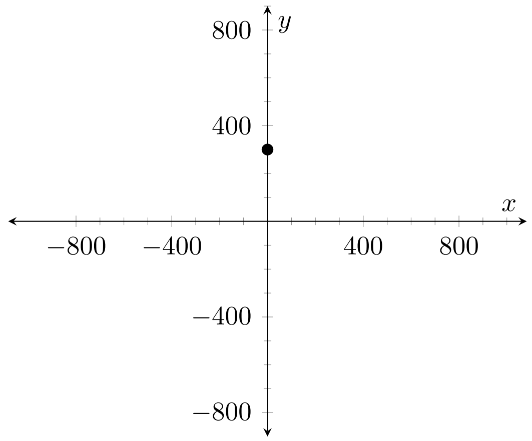
Solution: Here we take a look at a somewhat more difficult problem. The first step here will be to write the equation in slope-intercept form by solving for y.

$$\begin{aligned}
x - 3y + 900 &= 0 \\
x + 900 &= 3y && \text{Add } 3y \text{ to both sides} \\
\frac{x + 900}{3} &= \frac{3y}{3} && \text{Divide both sides by 3} \\
\frac{1}{3}x + \frac{900}{3} &= y \\
\frac{1}{3}x + 300 &= y && \text{Simplify fractions}
\end{aligned}$$

Next, clearly state the slope and y-intercept from the slope-intercept form of the equation from above. The slope here is already in fraction form.

$$\begin{aligned}
m &= \frac{1}{3} \\
b &= 300
\end{aligned}$$

Thirdly, we start with the y-intercept, remember that this is giving us a point on the y axis, specifically $(0, 300)$. Observing that we are not going to be able to use our standard scale. We use units of 100.

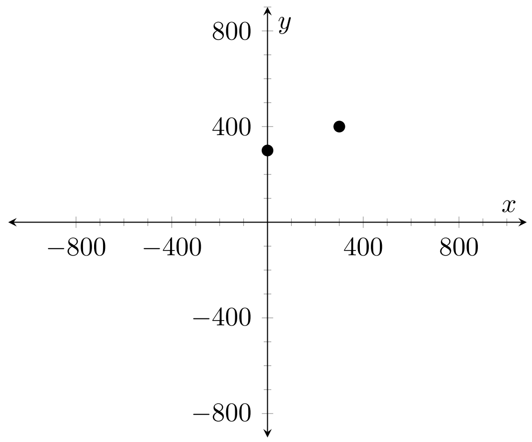


Now we have a problem with the slope in its current form. With the scale we have, it will be very difficult to measure up one unit and right 3 units (notice that the scale is in units of 100 per tick). To fix this problem we use something for our first few days in the course.

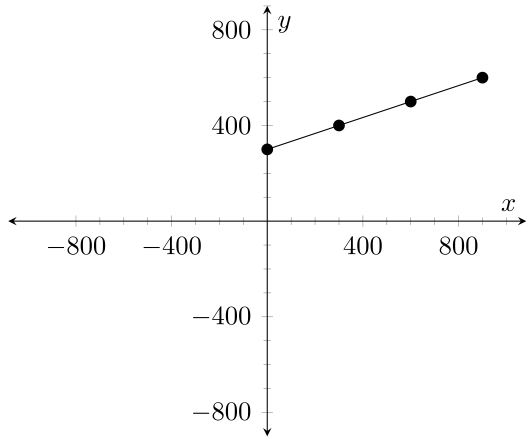
$$
\begin{aligned}
m &= \frac{1}{3} \\
&= \frac{1}{3} \times \frac{100}{100} && \text{We are multiplying by 1, which does not change the slope} \\
&= \frac{100}{300} && \text{This works well for our scale}
\end{aligned}
$$

The above work means that a slope of $\frac{1}{3}$ is the same as a slope of $\frac{100}{300}$. In other words, with respect to lines, we will still be on the same line using either slope, but clearly $\frac{100}{300}$ is easier to use with our scale.

Finally, we need another point to plot our line segment. So starting at the point $(0, 300)$ we go up 100 and right 300 and arrive at the point $(300, 400)$.



Notice that we are not finished yet because we need to know what the graph looks like up to $x = 900$. So we need points up to this x value. Further work using the slope to get new points will get us to $x = 900$. Giving us points at $(600, 500)$ and $(900, 600)$.



This concludes the example. ◀

IMPORTANT NOTICE

Students are recommended to study these previous examples and similar examples from the text book very carefully (remember this is section 4.2 in the textbook), as we will be using these to work problems in later chapters. Students that do not assimilate this material effectively run a considerable risk of low scores on Test 3.

4.1.4 The special cases of horizontal and vertical lines

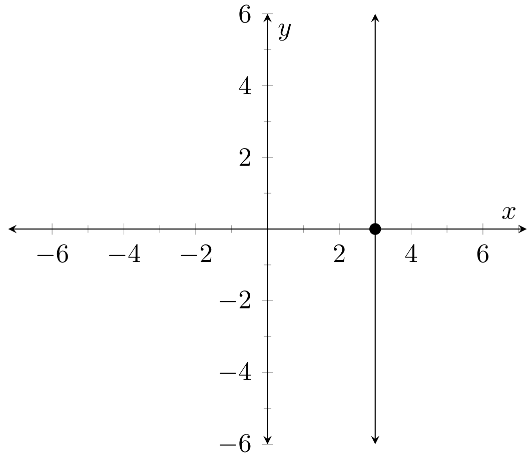
Students are advised to simply memorize the following two properties of horizontal and vertical lines. There is well established theory surrounding these mathematical phenomena, however they have little application for the beginning business student. Thus we will ignore these theoretical issues in class. Of course, if students would like to have a more detailed understanding of this material, then please consider seeing your instructor outside normal class time.

Vertical lines

Property 1. *Vertical lines have the property that their equation has the form $x = a$, where a is the x-intercept. The slope of all vertical lines is undefinied.*

Example 4.1.6. Graph $x = 3$.

Solution: The property above claims that the equation above is a vertical line and it has x-intercept at $x = 3$. There is only one line that satisfies this property.



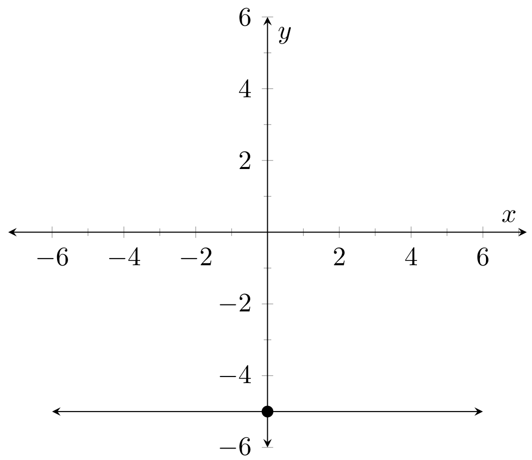
This concludes our problem. ◀

Horizontal lines

Property 2. *Horizontal lines have the property that their equation has the form $y = a$, where a is the y-intercept. The slope of all horizontal lines is zero.*

Example 4.1.7. Graph $y = -5$.

Solution: The property above claims that the equation above is a horizontal line and it has x-intercept at $x = -5$. There is only one line that satisfies this property.



Concluding our problem. ◀

4.2 Systems of Linear Equations

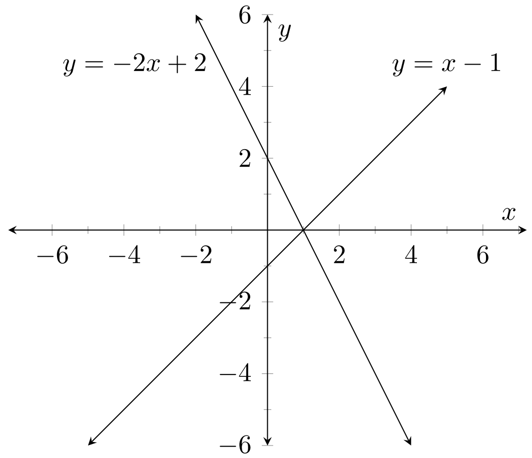
This section corresponds to section 4.1 in the textbook and is not officially part of the curriculum for this course. However, the methods in this section effectively replace the less precise geometric method used in section 4.3. The instructor will explain why this method is preferred in class. Students are recommended to work homework problems for section 4.1 in the textbook: part A 1-6 and part B 1-6 on pages 147 and 148. In section 4.3 students are recommended to do the same homework recommended by the course outline but should use the substitution method as described below. The geometric or graphical method is optional and students that choose to use this method are asked to tread carefully.

The primary goal of this section is in finding the intersection point of two lines. This process is called **Solving the System of Equations**. There are many methods that are both precise and unprecise. Some precise methods that you may be familiar with from previous mathematics courses are the Substitution method and Elimination method. A generalization of Elimination is called the Gauss-Jordan Reduction method, which makes use of matrices. There is also a hybrid of all of these methods called Gaussian Elimination. Only the substitution method is required for this course but students are welcome to use other methods, if they understand and can use them effectively. The Geometric method is the topic of section 4.3 of the textbook, which we discuss its weaknesses below.

Let us discuss the **Geometric Method** for a moment by considering the following two equations of a line:

$$\begin{cases} y = x - 1 \\ y = -2x + 2 \end{cases}$$

The geometric method is not difficult, we just need to use the methods from the previous section to generate graphs of each of the lines above on the same plane. In fact we have already graphed $y = x - 1$ at the beginning of subsection 4.1.3. Let us look at these two on the same graph.

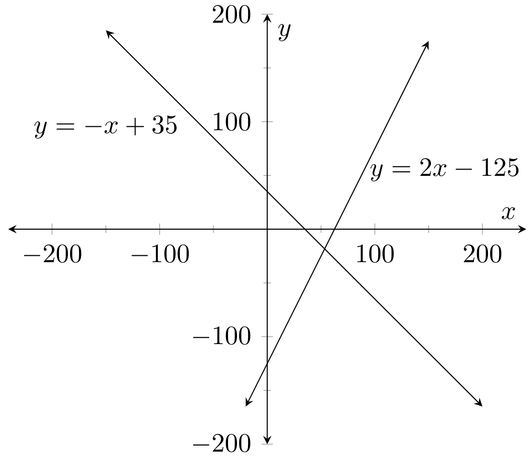


One can look and see that the intersection point is apparently at $(1, 0)$. In fact, we will show that this must be the intersection point soon.

Now consider another pair of lines.

$$\begin{cases} y = 2x - 125 \\ y = -x + 35 \end{cases}$$

We graph these lines for you below.



The question is: what is the intersection point? Looking at the picture, it looks like it might be near $(50, -25)$, but this is not clear at all. This example indicates the primary difficulty with the geometric method. Sometimes the exact solution (intersection point) is not clear from the picture. So the next question is: what can we do to remedy this?

4.2.1 Substitution method

The substitution method is an algebraic (not geometric) method for identifying the intersection point of two lines (provided such an intersection point exists). Students must decide for themselves how much detail is needed for each problem. We outline the method below.

Step 1: Choose one of the two equations, preferrably an equation that contains a variable without a coefficient.

Step 2: Choose a variable in the equation from step 1, preferrably a variable without a coefficient and even more preferably a variable that is already isolated.

Step 3: Solve the equation from step 1 for the variable chosen in step 2.

Step 4: Substitute the expression resulting from step 3 into the other equation.

Step 5: Solve the equation from step 4 for the remaining unknown (you should get a number).

Step 6: Substitute the number resulting from step 5 into any of the original equations or use the equation resulting from step 3. This will give you the number for the other unknown variable.

Step 7: Write the numbers you have found in step 5 and 6 as an order pair (remember the x goes first, then the y). This is our intersection point or solution.

Example 4.2.1. Find the intersection point of the system of equations

$$\begin{cases} y = x - 1 \\ y = -2x + 2 \end{cases}$$

(Remember we assumed that the intersection point was $(1, 0)$ above in a previous example, substitution method will allow us to check this result.)

Solution: We follow the steps above, however as you will see, many of the steps will be easy in this example:

Step 1: Choose one of the two equations, preferrably an equation that contains a variable without a coefficient.

Both equations have variables with no coefficent, so either equation will work for a choice here, so we choose $y = x - 1$, somewhat arbitrarily.

Step 2: Choose a variable in the equation from step 1, preferrably a variable without a coefficient and even more preferably a variable that is already isolated.

This is an easy choice looking at the equation chosen from step 1 because y is already isolated. So we choose the y variable.

Step 3: Solve the equation from step 1 for the variable chosen in step 2.

The equation is already solved for y. ($y = x - 1$)

Step 4: Substitute the expression resulting from step 3 into the <u>other</u> equation.

This gives us

$$x - 1 = -2x + 2$$

Step 5: Solve the equation from step 4 for the remaining unknown (you should get a number).

$$\begin{aligned} x - 1 &= -2x + 2 & & \\ 3x - 1 &= 2 & & \text{Add } 2x \text{ to both sides} \\ 3x &= 3 & & \text{Add 1 to both sides} \\ \frac{3x}{3} &= \frac{3}{3} & & \text{Divide both sides by 3} \\ x &= 1 & & \text{This is the } x \text{ value of the intersection point} \end{aligned}$$

Step 6: Substitute the number resulting from step 5 into any of the original equations or use the equation resulting from step 3. This will give you the number for the other unknown variable.

Here we choose to substitute $x = 1$ (from step 5) into the equation from step 3, giving us

$y = x - 1$	Equation from Step 3
$y = 1 - 1$	Substitute $x = 1$ from Step 5
$y = 0$	This is the y value of the intersection point

Step 7: Write the numbers you have found in step 5 and 6 as an order pair (remember the x goes first, then the y). This is our intersection point or solution.

So the intersection point or solution to the system is at $(1, 0)$, which verifies our estimation with the geometric method earlier concluding our example. ◀

Example 4.2.2. Find the intersection point of the system of equations

$$\begin{cases} y = 2x - 125 \\ y = -x + 35 \end{cases}$$

Solution: Again we follow the step by step process.

Step 1: Choose one of the two equations, preferrably an equation that contains a variable without a coefficient.

Both equations have varables with no coefficent, so either equation will work for a choice here, so we choose $y = -x + 35$, somewhat arbitrarily.

Step 2: Choose a variable in the equation from step 1, preferrably a variable without a coefficient and even more preferably a variable that is already isolated.

This is an easy choice looking at the equation chosen from step 1 because y is already isolated. So we choose the y variable.

Step 3: Solve the equation from step 1 for the variable chosen in step 2.

The equation is already solved for y. ($y = -x + 35$)

Step 4: Substitute the expression resulting from step 3 into the <u>other</u> equation.

This gives us

$$-x + 35 = 2x - 125$$

Step 5: Solve the equation from step 4 for the remaining unknown (you should get a number).

$$\begin{aligned} -x + 35 &= 2x - 125 && \text{Equation from step 4} \\ 35 &= 3x - 125 && \text{Add } x \text{ to both sides} \\ 160 &= 3x && \text{Add 125 to both sides} \\ \frac{160}{3} &= \frac{3x}{3} && \text{Divide both sides by 3} \\ 53.\overline{3} &= x && \text{This is the } x \text{ value for the intersection point} \end{aligned}$$

Step 6: Substitute the number resulting from step 5 into any of the original equations or use the equation resulting from step 3. This will give you the number for the other unknown variable.

Here we choose to substitute $x = 53.\overline{3}$ (from step 5) into the equation from step 3, giving us

$$\begin{aligned} y &= -x + 35 && \text{Equation from Step 3} \\ y &= -(53.\overline{3}) + 35 && \text{Substitute } x = 53.\overline{3} \text{ from Step 5} \\ y &= -18.\overline{3} && \text{This is the } y \text{ value of the intersection point} \end{aligned}$$

Step 7: Write the numbers you have found in step 5 and 6 as an order pair (remember the x goes first, then the y). This is our intersection point or solution.

So the intersection point or solution to the system is at $(53.33, -18.33)$ (we have rounded to two decimal places). This verifies our estimation with the geometric method from earlier. Notice, however, that our guess was off just a bit. Hopefully this makes it clear why the substitution method is better (it gives us exact answers). This concludes our example. ◀

Example 4.2.3. Solve the system of equations

$$\begin{cases} x = y \\ x - 3y + 3000 = 0 \end{cases}$$

Solution: Again, we use the process above.

Step 1: Choose one of the two equations, preferrably an equation that contains a variable without a coefficient.

Here we choose the equation $x = y$, because it is very simple and both variables have no coefficient

Step 2: Choose a variable in the equation from step 1, preferrably a variable without a coefficient and even more preferably a variable that is already isolated.

Interestingly, this equation is solved for both x and y.

Step 3: Solve the equation from step 1 for the variable chosen in step 2.

From step 2 the equation is already solved for either variable, $x = y$.

Step 4: Substitute the expression resulting from step 3 into the other equation.

Here we choose arbitrarily to substitute x in for y, however, the opposite can be done as well.

$$x - 3x + 3000 = 0$$

Step 5: Solve the equation from step 4 for the remaining unknown (you should get a number).

$x - 3x + 3000 = 0$	Equation from step 4
$-2x = -3000$	Combine like terms and subtract 3000 from both sides
$\frac{-2x}{-2} = \frac{-3000}{-2}$	Divide both sides by -2
$x = 1500$	This is the x value of our solution

Step 6: Substitute the number resulting from step 5 into any of the original equations or use the equation resulting from step 3. This will give you the number for the other unknown variable.

This is easy because we know that $x = y$. So $y = 1500$ as well.

Step 7: Write the numbers you have found in step 5 and 6 as an order pair (remember the x goes first, then the y). This is our intersection point or solution.

So the intersection point or solution to the system is at $(1500, 1500)$. This concludes our example. ◀

4.3 Applications-Cost-Volume-Profit Analysis

Students should be aware that this corresponds to section 6.1 in the textbook and in the course outline. Students should also be aware that this section will require significant preparation and homework, as it may be the most difficult section in the course.

Definition 11. *Let $R(x)$ and $C(x)$ be functions that describe the Total Revenue and Total Cost, respectively, of some business venture. Any solutions to the system of equations generated by these two functions is called a* **Break-Even point**. *The process of finding the break-even point and analyzing the geometry of the two functions $R(x)$ and $C(x)$ and how they relate to each other is called* **Cost-Volume-Profit Analysis**.

The textbook will use different notations for the functions (linear) discussed in the above definition. Instead of $R(x)$ we will use TR for total revenue, not to be confused with $T \times R$. Instead of $C(x)$ we will use TC for total cost.

4.3.1 Basic Break-Even Analysis Problems

Example 4.3.1. Rahmeen has had a computer repair business opened for a few months and has recorded some data relating to the income it makes and how much it costs to run. As far as costs are concerned he has to pay rent to the building owner and to an equipment company for using the space and rental of tools/equipment which comes to $900 per month and he has estimated that he has expenses associated with supplies and employee wages of $55 per computer repaired. As far as revenue, he charges, on average, $75 to repair each computer. Rahmeen has also determined that, at the very most, his company could repair 100 computers in one month. How many computers would Rahmeen's business have to service in order to break even on costs and revenues, given these estimations? The table below is a summary of some of the data above.

Computers Repaired	0	10	20	30	40	50	60
Revenue	$0	$750	$1500	$2250	$3000	$3750	$4500
Supplies/Materials	0	550	1100	1650	2200	2750	3300
Rent Cost	900	900	900	900	900	900	900
Total Costs	900	1450	2000	2550	3100	3650	4200
Net Income	-900	-700	-500	-300	-100	100	300

Table 4.1: Rahmeen's Summary for Up to 60 Computers Repaired in One Month

Solution: Our goal is to describe the data above in terms of a Total Cost function (TC) and a Total Revenue function (TR) and then to find where they are the same (find the intersection point of the two lines).

First, let us consider TR, the total revenue. The company's only revenue comes from the charges for repairing computers. He gets, on average, $75 per computer. So, if we let x be the number of computers repaired, then

$$TR = 75x \tag{4.1}$$

Next, let us look at TC, the total cost. There are two parts to the total cost of a business, variable cost VC and fixed cost FC. In other words,

$$TC = VC + FC \tag{4.2}$$

In our problem the variable cost is the costs that depend on the number of computers repaired, which is supplies/materials/employees, which costs, on average, $55 per computer repair. So $VC = 55x$. His fixed costs are those costs that do not change, which is rent. So $VC = 900$. In summary

$$TC = 55x + 900 \tag{4.3}$$

Thus we have our system of equations, TR and TC

$$\begin{cases} TR = 75x \\ TC = 55x + 900 \end{cases}$$

As an aside, we can also interprete this in the context of the previous section because we are trying to find the point where $TR = TC$

$$\begin{cases} y = 75x \\ y = 55x + 900 \end{cases}$$

Either of these interpretations may have value for our purposes.

So to solve the system we let

$$\begin{aligned} TR &= TC \\ 75x &= 55x + 900 \\ 20x &= 900 && \text{Subtract } 55x \text{ from both sides} \\ \frac{20x}{20} &= \frac{900}{20} && \text{Divide both sides by 20} \\ x &= 45 && \text{This is the } x \text{ value of the intersection point} \end{aligned}$$

We also need to find the y value of the intersection point. To do this we can substitute the x value back into either of the equations 4.1 or 4.3 above. We choose 4.1 here.

$$\begin{aligned} TR &= 75x \\ &= 75(45) && \text{From the work above} \\ &= 3375 && \text{This is the } y \text{ value of the intersection point} \end{aligned}$$

This number can be interpreted as the cost and revenue at the break even point.

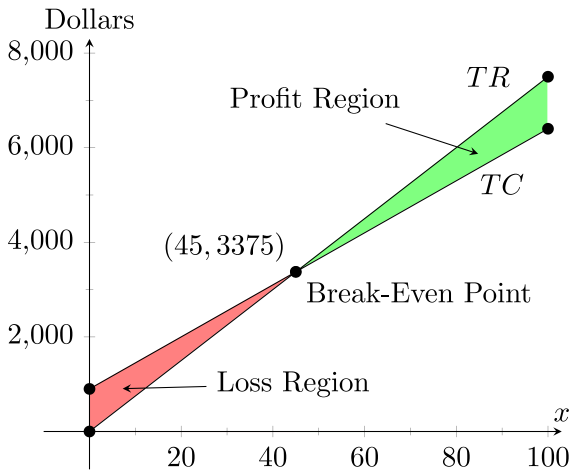
We will now discuss and draw a picture tieing all of this together. We will call this picture a **Break-Even Chart**. To draw our chart we will need to know a few points on each line (equations 4.1 and 4.3). We already know a few obvious points. For example, the revenue function has a point at $(0, 0)$, this is its y-intercept (see slope intercept form). We also know that the break even point is on this graph, $(45, 3375)$. The cost function also has a few obvious points. Its y-intercept is at $(0, 900)$ and the break even point is also on this line.

There are a pair of less obvious, but important points that we should know before drawing our graph. We should always know the **cost and revenue at capacity**. This is found by letting x in each function be its maximum practical or possible value. For our example we are told that Rahmeen has determined that his company can repair at most 100 computers in a month, so let $x = 100$ and substitute into each of TR and TC.

$$\begin{aligned} TR &= 75(100) = 7500 \\ TC &= 55(100) + 900 = 6400 \end{aligned}$$

Thus we have two more points to include. $(100, 7500)$ on the line for TR and $(100, 6400)$ on the line for TC. Now let us summarize before we draw our chart.

	Points to be Graphed
TR	(0,0) (45,3375) (100,7500)
TC	(0,900) (45,3375) (100,6400)



Notice here that we have shaded certain parts of the graph. The shaded region to the left of the break-even point is called the **Loss Region** because here costs are more than revenues. Conversely, the region to the right of the break-even point is called the **Profit Region** because revenue is more than costs. This concludes our first Cost-Volume Profit Analysis problem. ◀

Steps for Finding the Break-Even Point

Step 1 Read the problem very carefully, perhaps two or more times, before doing any writing.

Step 2 Find TR, there is only one component to TR, and it will usually depend on x. Look for words like: income or revenue. In the problem above, Rahmeen was charging his customers on average \$75 per computer repair. This is giving us information about TR.

Step 3 Find TC. Remeber $TC = VC + FC$, in other words there are usually two components to total cost. VC will always depend on x, FC will be constant.

Step 4 Solve the system of equations resulting from TR and TC, by setting $TR = TC$, to find the Break-Even point. Remember to find both the x and the y values.

Steps for Drawing the Break-Even Chart

Step 1 Find the y-intercepts of both TR and TC (see slope-intercept form). Make sure that you are clear as to what point is on which line.

Step 2 Find the cost and revenue at capacity. Often this can be found from an x value (at capacity), given to you in the problem, that you can substitute into TR and TC for another point on each of these lines.

Step 3 Draw the graph with just the scales on the axes. Your scales do not have to be the same on both axes, in fact most of the time they will be different. The maximum value on your x axis will need to be at or a little more than the x value given at capacity. The maximum value on your y axis will need to be at or a little more than the y value at capacity on TR (see step 2). Place intermediate ticks on the scale all the way back to zero and label them uniformly.

Step 4 Plot your TR points generated from the steps above and remember, the break-even point from the previous step-by-step process is a point on both TR and TC. Draw a line to connect these points on TR.

Step 5 Plot your TC points from above. Draw a line to connect these points. By now you should see that your TR and TC cross over each other at the break-even point.

Step 6 Label the break-even point (with coordinates) and the loss and profit regions.

Example 4.3.2. Stephanie is a factory manager for a corporation that makes products. Stephanie has determined that the corporation incurs fixed costs of \$200 per week on property taxes and various upkeep. The corporation also has to pay the employees on average \$980 per employee per week by union contract. After discussing things with her CFO, she has determined that each employee is contributing about \$1000 to corporate revenue, further she knows that due to the small size of the factory she can have at most 20 workers on any given week. What is the minimum number of employees per week needed to keep this corporation profitable? Express the number of employees at the break-even point as a percent of capacity.

Solution: After reading the problem carefully we need to construct TR. We know from the problem that each employee is contributing revenue to the corporation to the tune of \$1000 per employee. So if we let x be the number of employees, then

$$TR = 1000x \tag{4.4}$$

Secondly, for TC. We know that there is a fixed tax expense of \$200 ($FC = 200$) and a variable cost of \$980 per employee for wages ($980x$). So

$$\begin{aligned} TC &= VC + FC \\ &= 980x + 200 \end{aligned} \tag{4.5}$$

Now that we have TR and TC we can find the break even point by letting $TR = TC$.

$$\begin{aligned} TR &= TC \\ 1000x &= 980x + 200 \\ 20x &= 200 && \text{Subtract 980 from both sides} \\ \frac{20x}{20} &= \frac{200}{20} && \text{Divide both sides by 20} \\ x &= 10 && \text{This is our } x \text{ value at the break even point} \end{aligned}$$

Now recall that the break even point involves neither loss nor profit. Note that the question in the problem was what is the minimum number of employees required to be profitable. Thus we would need at least 11 employees to be profitable.

To express the break-even point as a percent of capacity we divide

$$\frac{x \text{ value at break-even point}}{x \text{ value at capacity}} \tag{4.6}$$

or

$$\frac{y \text{ value at break-even point}}{y \text{ value at capacity}} \tag{4.7}$$

depending on what is given in the problem. We use the first expression here.

$$\frac{x \text{ value at break-even point}}{x \text{ value at capacity}} = \frac{10}{20} = 0.5 = 50\%$$

Concluding our problem. ◀

Example 4.3.3. Draw a break-even chart for the given cost and revenue functions. Assume that $x = 25$ is capacity.

$$\begin{cases} TC = x + 15 \\ TR = 3x \end{cases}$$

Solution: We begin with the fact that the y-intercepts are $(0, 15)$ for TC and $(0, 0)$ for TR. The break-even point is found by setting TR and TC equal:

$$\begin{aligned} TR &= TC \\ 3x &= x + 15 \\ 2x &= 15 && \text{Subtract } x \text{ from both sides} \\ \frac{2x}{2} &= \frac{15}{2} && \text{Divide both sides by 2} \\ x &= 7.5 \end{aligned}$$

To find the y-value, we substitute this into TR.

$$\begin{aligned} TR &= 3x \\ &= 3(7.5) \\ &= 22.5 \end{aligned}$$

Thus the break-even point is at $(7.5, 22.5)$.

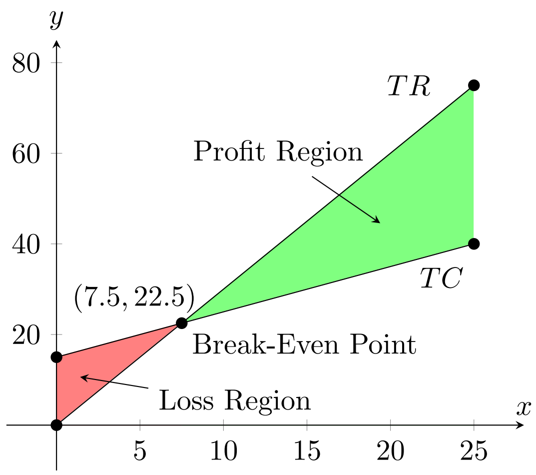
We also need the y-values at capacity. We can find this by substituting $x = 25$ into both TR and TC.

$$TC = x + 15 = 25 + 15 = 40$$
$$TR = 3x = 3(25) = 75$$

We have more points: $(25, 40)$ on TC and $(25, 75)$ on TR.

Finally, we have a summary of all the points and the chart.

	Points to be Graphed
TR	(0,0) (7.5,22.5) (25,75)
TC	(0,15) (7.5,22.5) (25,40)



and that finishes the problem. ◀

4.3.2 Other Break-Even Analysis Problems

The next three problems are generally optional within BMAT110 and should be considered more challenging.

Example 4.3.4. The following information is available about a corporation for the current year. Sales \$40000, Fixed costs, \$12600, Variable costs \$16000 and Capacity at sales maximum of \$60000. Find the total revenue and total cost functions. Find the break even point in sales dollars and as a percent of capacity.

Solution: Our first objective is to find the Total Revenue (TR) and Total Cost (TC) functions. In constructing TR and TC we will let x be the the total dollar value of the product sold. We do this because we do not know the price per unit or cost per unit from the problem. Thinking about this carefully gives us:

$$TR = x$$

Now to find TC. We know that $TC = VC + FC$, where VC is Variable Costs and FC is Fixed Costs. We know from the problem that $FC = 12600$. To find the variable cost we need to consider how many cents of every dollar of revenue goes to variable costs. To find this compute:

$$\frac{\text{Total Variable Cost}}{\text{Total Revenue}} = \frac{16000}{40000} = 0.4 = 40\%$$

Computed from the data in the problem. This tells us that for every dollar of money incoming, 40 cents must go to variable costs. Further, since x is the dollar value of the units sold:

$$VC = 0.4x$$

Giving us:

$$\begin{aligned} TC &= VC + FC \\ &= 0.4x + 12600 \end{aligned}$$

So we have our Total Revenue and Total Cost Functions (parts 1a and 1b in the problem). On to part 2.

From this point on the problem is exactly like all of the other Break Even problems we have done. We now need to do the break even analysis. Part (a) wants us to find the break even point in terms of sales dollars. To do this set $TR = TC$ and solve.

$$\begin{aligned} TR &= TC \\ x &= 0.4x + 12600 \\ 1x &= 0.4x + 12600 \\ 1x - 0.4x &= 12600 \\ 0.6x &= 12600 \\ x &= \frac{12600}{0.6} \\ x &= 21000 \end{aligned}$$

This tells us that the break even point in terms of dollars is \$21 000.

Next in part (2b), we need to find the break-even point as a percent of capacity. This is the easy part. To find this simply compute:

$$\begin{aligned} \frac{\text{Break Even Point in Terms of Dollars}}{\text{Sales Value at Capacity}} &= \frac{21000}{60000} \\ &= 0.35 \\ &= 35\% \end{aligned}$$

Notice that from the orginal problem it says that the capacity is at sales maximum of \$60 000. This is how we know "Sales Value at Capacity" in the formula above. ◀

Example 4.3.5. The following data pertain to the operating budget of Matt Manufacturing. Capacity per period is sales of \$800,000. In a period Sales were \$720,000, Fixed Costs were \$220,000, Total Variable Cost was \$324,000. Find the Total Cost Function, Total Revenue Function, Break even point in terms of dollars, and Break even point as a percent of Capacity.

Solution: This will all be very similar to the previous problem. Our first objective is to find the Total Revenue (TR) and Total Cost (TC) functions. In constructing TR and TC we will let x be the the total dollar value of the product sold. We do this because we do not know the price per unit or cost per unit from the problem. Thinking about this carefully gives us:

$$TR = x$$

Now to find TC. We know that $TC = VC + FC$, where VC is Variable Costs and FC is Fixed Costs. We know from the problem that $FC = 220000$. To find the variable cost we need to consider how many cents of every dollar of revenue goes to variable costs. To find this compute:

$$\frac{\text{Total Variable Cost}}{\text{Total Revenue}} = \frac{324000}{720000} = 0.45 = 45\%$$

Computed from the data in the problem. This tells us that for every dollar of money incoming, 45 cents must go to variable costs. Further, since x is the dollar value of the units sold:

$$VC = 0.45x$$

Giving us:

$$\begin{aligned} TC &= VC + FC \\ &= 0.45x + 220000 \end{aligned}$$

So we have our Total Revenue and Total Cost Functions.

From this point on the problem is exactly like all of the other Break Even problems we have done. We now need to do the break even analysis. We will now find the break even point in terms of sales dollars. To do this set $TR = TC$ and solve.

$$\begin{aligned} TR &= TC \\ x &= 0.45x + 220000 \\ 1x &= 0.45x + 220000 \\ 1x - 0.45x &= 220000 \\ 0.55x &= 220000 \\ x &= \frac{220000}{0.55} \\ x &= 400000 \end{aligned}$$

This tells us that the break even point in terms of dollars is \$400 000.

Next, to find the break-even point as a percent of capacity. This is the easy part. To find this simply compute:

$$\begin{aligned} \frac{\text{Break Even Point in Terms of Dollars}}{\text{Sales Value at Capacity}} &= \frac{400000}{800000} \\ &= 0.50 \\ &= 50\% \end{aligned}$$

Notice that from the orginal problem it says that the capacity is at sales maximum of \$800 000. ◀

Example 4.3.6. Harrow Seed and Fertilizer compiled the following estimates for its operations. Capacity per period is sales of \$150,000. Sales during a specific period were \$120,000, Fixed Costs were \$43,200, Total Variable Cost was \$48,000. Find the Total Cost Function, Total Revenue Function, Break even point in terms of dollars, and Break even point as a percent of Capacity.

Solution: Our first objective is to find the Total Revenue (TR) and Total Cost (TC) functions. In constructing TR and TC we will let x be the the total dollar value of the product sold. We do this because we do not know the price per unit or cost per unit from the problem. Thinking about this carefully gives us:

$$TR = x$$

Now to find TC. We know that $TC = VC + FC$, where VC is Variable Costs and FC is Fixed Costs. We know from the problem that $FC = 43200$. To find the variable cost we need to consider how many cents of every dollar of revenue goes to variable costs. To find this compute:

$$\frac{\text{Total Variable Cost}}{\text{Total Revenue}} = \frac{48000}{120000} = 0.4 = 40\%$$

Computed from the data in the problem. This tells us that for every dollar of money incoming, 40 cents must go to variable costs. Further, since x is the dollar value of the units sold:

$$VC = 0.4x$$

Giving us:

$$\begin{aligned} TC &= VC + FC \\ &= 0.4x + 43200 \end{aligned}$$

So we have our Total Revenue and Total Cost Functions.

From this point on the problem is exactly like all of the other Break Even problems we have done. We now need to do the break even analysis. We will now find the break even point in terms of sales dollars. To do this set $TR = TC$ and solve.

$$\begin{aligned} TR &= TC \\ x &= 0.4x + 43200 \\ 1x &= 0.4x + 43200 \\ 1x - 0.4x &= 43200 \\ 0.6x &= 43200 \\ x &= \frac{43200}{0.6} \\ x &= 72000 \end{aligned}$$

This tells us that the break even point in terms of dollars is \$72 000.

Next, to find the break-even point as a percent of capacity. This is the easy part. To find this simply compute:

$$\begin{aligned} \frac{\text{Break Even Point in Terms of Dollars}}{\text{Sales Value at Capacity}} &= \frac{72000}{150000} \\ &= 0.48 \\ &= 48\% \end{aligned}$$

Notice that from the orginal problem it says that the capacity is at sales maximum of \$150 000. ◄

This concludes chapter 4.

5 Discount Series and Markup

In this chapter we take a deeper look at the topics of discount and markup. Each of these topics have been discussed at length in previous chapters. It might be a good idea to review sections that involve percentage decrease and increase, which can be found in chapter 3.

5.1 Discount Series

5.1.1 Net price factor method

You may recall working several discount problems earlier in the course. Among these problems you may have encountered several formulae

$$\text{Amount of Discount} = \text{List Price} \times \text{Rate of Discount}$$

or

$$A = L \times d \tag{5.1}$$

From this equation we can derive other formulae:

$$L = \frac{A}{d} \tag{5.2}$$

$$d = \frac{A}{L} \tag{5.3}$$

You may recall that the above formulae are used to compute the amount of discount and that if we wish to compute to new price we have to subtract the amount of discount from the list price. In mathematics

$$N = L - Ld \qquad \text{where } N \text{ is the new price} \tag{5.4}$$

This is a completely useful form for computing a discount provided there is only one discount involved. Thus a natural question is: what happens if there is more than one discount on a list price? We call such a situation a **Discount Series**, and is the primary topic of this section.

To answer the question above with formula 5.4 we would have to do a lot of complicated arithmetic. In fact, the arithmetic would become significantly more complicated as we add more discounts. However, there is a nice alternative. We call this alternative the **Factor Form** of equation 5.4 and now derive it.

$$\begin{aligned} N &= L - Ld \\ &= L \underbrace{(1-d)}_{\text{net price factor of } d \text{ (NPF)}} && \text{Factor out } L \end{aligned} \tag{5.5}$$

Equation 5.5 is the net factor equivalent of equation 5.4.

This might not seem very significant and a bit anti-climactic, but let us consider an example to motivate this a bit further.

Example 5.1.1. Using equation 5.5, find the price paid at the cashier on an item with a list price of \$69.99 with a discount of 10%.

Solution: Here N is the value we are looking for, while $d = 0.10$, and $L = 69.99$. So

$$\begin{aligned} N &= L(1-d) \\ &= 69.99(1-0.10) \\ &= 69.99(0.90) \\ &= 62.991 \\ &= \$62.99 \end{aligned}$$

So our answer is \$62.99. ◀

This should seem easier than using equation 5.4. It turns out it is far easier when dealing with more than one discount.

Property 3. *Let some list price L be subject to the discount series d_1, d_2, d_3 $\ldots d_n$, then the net price N is*

$$N = L \underbrace{(1-d_1)}_{\textit{NPF of discount 1}} \overbrace{(1-d_2)}^{\textit{NPF of discount 2}} \underbrace{(1-d_3)}_{\textit{NPF of discount 3}} \ldots \overbrace{(1-d_n)}^{\textit{NPF of discount n}} \tag{5.6}$$

Example 5.1.2. Find the net price of a list price total of \$1203, subject to the discount series 7%, 5%, and 3%. Round your answer to the nearest cent.

Solution: Here we proceed with property 3.

$$\begin{aligned} N &= L(1-d_1)(1-d_2)(1-d_3) \\ &= 1203(1-0.07)(1-0.05)(1-0.03) \\ &= 1203(0.93)(0.95)(0.97) \\ &= 1030.964985 \\ &= \$1030.96 \end{aligned}$$

So our net price is \$1030.96. ◀

Example 5.1.3. Eli's company makes windows and purchases materials from a local company, owned by Herb, who Eli has known for many years. As a result of the close relationship Eli receives discounts on the material he purchases. First, Eli always purchases in bulk which gives him a discount of 9%. Secondly, his workshop is nearby which effectively amounts to a discount for shipping of 6%. Lastly, since Eli and the materials company owner are good friends he gets a 2% discount. If Eli purchases materials totaling $4,832, how much will he actually pay?

Solution: Here, again, we proceed with property 3.

$$\begin{aligned} N &= L(1-d_1)(1-d_2)(1-d_3) \\ &= 4832(1-0.09)(1-0.06)(1-0.02) \\ &= 4832(0.91)(0.94)(0.98) \\ &= 4050.626944 \\ &= \$4050.63 \end{aligned}$$

So he pays $4050.63. ◀

5.1.2 Single equivalent rate of discount

It is sometimes useful to understand that any discount series can be interpreted as a single discount or single equivalent rate of discount. We find this with the following definition.

Definition 12. *Let d_1, d_2, $d_3 \ldots d_n$ be some discount series of n discounts, then the* **Single Equivalent Rate of Discount** *is*

$$\begin{aligned} \textit{Single Equivalent Rate of Discount} &= 1-(1-d_1)(1-d_2)(1-d_3)\ldots(1-d_n) \\ &= 1 - \textit{Product of the NPF's} \end{aligned}$$

Example 5.1.4. Find the single equivalent rate of discount in example 5.1.2. The discount series was 7%, 5%, and 3%. Round to the nearest percent.

Solution: From defintion 12 above

$$\begin{aligned} \text{Single Equivalent Rate of Discount} &= 1-(1-d_1)(1-d_2)(1-d_3) \\ &= 1-(1-0.07)(1-0.05)(1-0.03) \\ &= 1-(0.93)(0.95)(0.97) \\ &= 0.143005 \\ &= 14\% \end{aligned}$$

So the single equivalent rate of discount is 14%. ◀

Example 5.1.5. Find the single equivalent rate of discount in example 5.1.3. The discount series was 9%, 6%, and 2%. Round to the nearest percent.

Solution: From definition 12

$$\begin{aligned}\text{Single Equivalent Rate of Discount} &= 1-(1-d_1)(1-d_2)(1-d_3)\\ &= 1-(1-0.09)(1-0.06)(1-0.02)\\ &= 1-(0.91)(0.94)(0.98)\\ &= 0.161708\\ &= 16\%\end{aligned}$$

So the single equivalent rate of discount is 16%. ◀

This final example is meant to be somewhat of a challenge, but could show up on an exam or quiz.

Example 5.1.6. Referencing example 5.1.3 with Eli's company, consider a situation where a competitor company comes in and offers Eli a single discount of 17% on purchases, if he will buy from them. If Herb wants to keep Eli's business he must match the new company's discount. What fourth discount should Herb provide? Assume Eli is purchasing $5000 worth of materials and round the answer to the nearest percent.

Solution: Here we will use property 3 and note that we want the net price from the two business to be the same and note that we need a variable to find the fourth discount from Herb's business. We need to find this d_4.

$$\begin{aligned}L(1-d_1)(1-d_2)(1-d_3)(1-d_4) &= L(1-d_{\text{competitor}})\\ 5000(1-0.09)(1-0.06)(1-0.02)(1-d_4) &= 5000(1-0.17)\\ 5000(0.91)(0.94)(0.98)(1-d_4) &= 5000(0.83) && \text{Compute NPF's}\\ 4191.46(1-d_4) &= 4150 && \text{Multiply}\\ \frac{4191.46(1-d_4)}{4191.46} &= \frac{4150}{4191.46} && \text{Divide both sides by 4191.46}\\ 1-d_4 &= 0.990108458\\ 1-0.990108458 &= d_4 && \text{Isolate } d_4\\ 0.00989154\ldots &= d_4\\ 1\% &= d_4 && \text{Round after multiplying by 100}\end{aligned}$$

So Herb would have to include an additional discount of 1%. ◀

5.2 Markup

This section corresponds to section 5.3 in the textbook.

Here we are interested in how to precisely markup products for sale that have been purchased from a third party after all expenses and costs are taken into account. Alternatively, we might be purchasing many products to assemble then sale.

5.2.1 Some basic concepts

Definition 13. *Let c be the cost to buy some product, e be the expenses or overhead of the business, and p be the profit required by the business owner; then the* **selling price** *of the product is*

$$s = c + e + p \tag{5.7}$$

We can manipulate this equation algebraically to get some other useful formulae.

$$s - c = e + p \tag{5.8}$$

We sometimes call the expense added to profit the markup, m.

$$m = e + p \tag{5.9}$$

$$s = c + m \tag{5.10}$$

Example 5.2.1. Vlad owns a wood shop and wants to sell cut lumber for \$600 per ton to compete with other businesses in the area. After some research he has determined that he can purchase the lumber from a mill at a cost of \$350 per ton and his overhead will amount to about 15% of the cost. Under these circumstances, how much profit can he make?

Solution: Here we use equation 5.7 with $s = 600$, $c = 350$, $e = 0.15c$, we are looking for p. Putting all of this together we have.

$$\begin{aligned} s &= c + e + p \\ 600 &= 350 + 0.15(350) + p && \text{Substitute in } c \text{ and } e \\ 600 &= 402.5 + p \\ \$197.50 &= p && \text{Subtract 402.5 from both sides} \end{aligned}$$

So our answer is \$197.50. ◀

Example 5.2.2. Nihkita has two options to fill a spot on a shelf in her store. Item A costs \$59 and sells for \$99. Item B costs \$52 and sells for \$102. Overhead of the business is 20% of the cost. Determine the markup, expense, and profit for each item. Which product would be better to have on the shelf in terms of profit?

Solution: We start with equation 5.10 to find the markup and use subscripts to denote the item for each calculation.

$$\begin{aligned} s_a &= c_a + m_a \\ 99 &= 59 + m_a && \text{Substitute in } c \text{ and } s \\ \$40 &= m_a \end{aligned}$$

$$\begin{aligned} s_b &= c_b + m_b \\ 102 &= 52 + m_b && \text{Substitute in } c \text{ and } s \\ \$50 &= m_b \end{aligned}$$

Now for the expense. We know that the overhead (expense) is 20% of the cost. or $e = 0.20c$.

$$\begin{aligned} e_a &= 0.20c_a \\ &= 0.20(59) \\ &= \$11.80 \end{aligned}$$

$$\begin{aligned} e_b &= 0.20c_b \\ &= 0.20(52) \\ &= \$10.40 \end{aligned}$$

Finally, from equation 5.9 we know that $p = m - e$.

$$\begin{aligned} p_a &= m_a - e_a \\ &= 40 - 11.80 \\ &= \$28.20 \end{aligned}$$

$$\begin{aligned} p_b &= m_b - e_b \\ &= 50 - 10.40 \\ &= \$39.60 \end{aligned}$$

Item B would be better to have on the shelf because it will generate more profit. ◀

5.2.2 Rate of markup

Markup is usually interpreted in two different ways:

1. As a percent of cost.
2. As a percent of selling price.

So we have two different defintions:

Definition 14. *Let m be the markup, c be the cost, s be the selling price, r_c be the* **rate of markup based on cost**, *and r_s be the* **rate of markup based on selling price**, *then*

$$r_s = \frac{m}{s} \tag{5.11}$$

and

$$r_c = \frac{m}{c} \tag{5.12}$$

Example 5.2.3. Compute (a) the missing value (cost, selling price, or markup); (b) the rate of markup based on cost; and (c) the rate of markup based on selling price for each of the following:

(i) Cost is \$50; selling price is \$75

(ii) Selling price is \$95; markup is \$15

(iii) Cost is \$60; markup is \$8

Solution: In each problem we first need to find the missing value using equation 5.10 on page 77, then substitute into the appropriate formula from defintion 14 above.

(i) In this problem the missing value is the markup. So

$$\begin{aligned} s &= c + m \\ 75 &= 50 + m \\ 25 &= m && \text{Subtract 50 fom both sides} \end{aligned}$$

Now to find r_s and r_c.

$$\begin{aligned} r_s &= \frac{m}{s} \\ &= \frac{25}{75} \\ &= 0.\overline{33} \\ &\approx 33\% \end{aligned}$$

and

$$\begin{aligned} r_c &= \frac{m}{c} \\ &= \frac{25}{50} \\ &= 0.5 \\ &= 50\% \end{aligned}$$

(ii) Here the missing value is cost. So

$$\begin{aligned} s &= c + m \\ 95 &= c + 15 \\ 80 &= c && \text{Subtract 15 fom both sides} \end{aligned}$$

Now to find r_s and r_c.

$$\begin{aligned} r_s &= \frac{m}{s} \\ &= \frac{15}{95} \\ &= 0.1578947\ldots \\ &\approx 16\% \end{aligned}$$

and

$$\begin{aligned} r_c &= \frac{m}{c} \\ &= \frac{15}{80} \\ &= 0.1875 \\ &\approx 19\% \end{aligned}$$

(iii) Here the missing value is selling price. So

$$\begin{aligned} s &= c + m \\ &= 60 + 8 \\ &= 68 \end{aligned}$$

Now to find r_s and r_c.

$$\begin{aligned} r_s &= \frac{m}{s} \\ &= \frac{8}{68} \\ &= 0.117647\ldots \\ &\approx 12\% \end{aligned}$$

and

$$\begin{aligned} r_c &= \frac{m}{c} \\ &= \frac{8}{60} \\ &= 0.1\overline{3} \\ &\approx 13\% \end{aligned}$$

This concludes our problem and this chapter of the course. ◀

6 Simple Interest

This is the final chapter in the course and corresponds to Chapter 7 in the textbook.

This chapter is designed to give the reader the most basic idea behind interest calculations. While you are reading through this material understand that in practical application simple interest is not used except over the short term. In the medium and long term, because of the compounding nature of interest, we must use more complex formulae which make use of more complex algebra (i.e. Exponential and Logarithmic Equations) which is outside the scope of this course. Those students that go on to take a Mathematics of Finance course will encounter Compounding Interest and the mathematics behind it. It is very important for students to remember that the formulae throughout this chapter only apply to simple interest and not compound interest.

6.1 The Simple Interest Formula

Definition 15. Interest *is the fee charged for the use of borrowing money. If I is the amount of* **simple interest**, *P is the principal or original amount borrowed, r is the rate of interest per annum (yearly or nominal rate of interest) expressed as a percent which should be converted to decimal form, and t is the length of time the money is borrowed in units of years (sometimes called the period of the loan); then*

$$I = Prt$$

Example 6.1.1. Find the amount of interest needed to pay off a loan with a principal of \$10,000 over a period of 2 years at an interest rate of 8%.

Solution: Use the equation from definition 15.

$$\begin{aligned} I &= Prt \\ &= 10000(0.08)(2) \\ &= \$1600 \end{aligned}$$

So the amount of interest will be \$1600. ◀

Example 6.1.2. Find the amount of interest needed to pay off a loan with a principal of \$5000 over a period of 18 months at an interest rate of 4%.

Solution: Use the equation from definition 15 and remember to convert 18 months to units of years by dividing by 12.

$$\begin{aligned} I &= Prt \\ &= 5000(0.04)\left(\frac{18}{12}\right) \\ &= \$300 \end{aligned}$$

So the amount of interest to pay off the loan would be \$300. ◀

Example 6.1.3. Find the amount of interest needed to pay off a loan with a principal of \$950 over a period of 130 days at an interest rate of 3.25%.

Solution: Use the equation from definition 15 and remember to convert 130 days to units of years by dividing by 365.

$$\begin{aligned} I &= Prt \\ &= 950(0.0325)\left(\frac{130}{365}\right) \\ &= \$11 \end{aligned}$$

We would need \$11 to pay off the loan. ◀

6.2 Finding Principal, Rate, or Time

This section corresponds to section 7.2 in the textbook.
We also need to be able to find the principal, rate, or time in similar problems. In order to do this we just need to manipulate the equation from definition 15 in the previous section.

1. To find the principal, we solve for P in the equation from definition 15.

$$\begin{aligned} I &= Prt \\ \frac{I}{rt} &= \frac{Prt}{rt} && \text{Divide both sides by } rt \\ \frac{I}{rt} &= P && \text{This allows us to compute } P \qquad (6.1) \end{aligned}$$

2. To find the rate, we solve for r in a similar fashion as above using the equation from definition 15. We arrive at

$$r = \frac{I}{Pt} \tag{6.2}$$

3. Finally, to find the time, we solve for t again similar to part 1.

$$t = \frac{I}{Pr} \tag{6.3}$$

Example 6.2.1. Find the time needed to accumulate \$1500 on a principal of \$10,000 at an interest rate of 10%

Solution: Here we use equation 6.3 above because we are looking for time.

$$\begin{aligned} t &= \frac{I}{Pr} \\ &= \frac{1500}{10000 \times 0.10} \\ &= 1.5 \text{ years} \end{aligned}$$

So we would need to wait 1.5 years to accumulate \$1500. ◀

Example 6.2.2. Find the interest rate needed to accumulate interest of \$1200 on a principal of \$30,000 over 10 months

Solution: Here we use equation 6.2 above because we are looking for r. Remember to conver 10 months to years by dividing by 12.

$$\begin{aligned} r &= \frac{I}{Pt} \\ &= \frac{1200}{30000 \times 10/12} \\ &= 0.048 \\ &= 4.8\% \end{aligned}$$

So the interest rate would have to be 4.8%. ◀

Example 6.2.3. Find the principal needed to build up interest of \$10,000 over 400 days at an intest rate of 13%.

Solution: Here we use equation 6.1 above because we are looking for P. Make sure to convert days to years.

$$\begin{aligned} P &= \frac{I}{rt} \\ &= \frac{10000}{0.13 \times 400/365} \\ &= 70192.30769\ldots \\ &= \$70,192.31 \end{aligned}$$

So the principal would need to be \$70,192.31. ◀

6.3 Future or Maturity Values

Definition 16. *The* **Future Value** *of an amount of money (or* **Maturity Value***), S, is the value obtained after adding the principal to the interest. So*

$$S = P + I$$

Unfortunately, this form is not very useful for applications, so we will do a bit of algebra on it.

$$\begin{aligned} S &= P + I \\ &= P + Prt && \text{From definition 15} \\ &= P(1 + rt) && \text{Factor out } P \end{aligned} \tag{6.4}$$

Example 6.3.1. Find the future value of an investment of \$675 at 4% over 14 months.

Solution: Here we use equation 6.4. Do not forgot to convert months to years.

$$\begin{aligned} S &= P(1 + rt) \\ &= 675\left(1 + 0.04\left(\frac{14}{12}\right)\right) \\ &= \$706.50 \end{aligned}$$

So the future value is \$706.50. ◀

Example 6.3.2. Find the maturity value of a deposit into a government insured bond valued at \$7000, for 300 days, at an interest rate of 2.8%.

Solution: Here we use equation 6.4 again. Do not forgot to convert days to years.

$$\begin{aligned} S &= P(1 + rt) \\ &= 7000\left(1 + 0.028\left(\frac{300}{365}\right)\right) \\ &= 7161.09589\ldots \\ &= \$7161.10 \end{aligned}$$

So the maturity value is \$7161.10. ◀

6.4 Finding the Principal or Present Value

In many applications we are interested in finding the principal (sometimes called the present value). For example, in a retirement scenario, we might wish to find out how much we need to initially invest to accumulate a certain amount of money at a certain rate over a period of time. This problem can be delt with by using equation 6.4 or doing some algebra on this equation and then just substitute into the newly derived equation. We derive that equation next and allow the student to choose how they wish to solve the problems. We will use the equation derived below.

Starting with equation 6.4

$$
\begin{aligned}
S &= P(1+rt) \\
\frac{S}{1+rt} &= \frac{P(1+rt)}{1+rt} && \text{Divide both sides by } 1+rt \\
\frac{S}{1+rt} &= P && (6.5)
\end{aligned}
$$

Example 6.4.1. Find the initial value of an investment which is valued at \$8345 at maturity and earned interest at 4.25% per annum (p.a.) over 9 months.

Solution: Here we use equation 6.5. Do not forgot to convert months to years and be careful with the order of operations.

$$
\begin{aligned}
P &= \frac{S}{1+rt} \\
&= \frac{8345}{1+0.0425\times 9/12} \\
&= 8087.21986\ldots \\
&= \$8087.22
\end{aligned}
$$

So the initial value or principal would need to be \$8087.22. ◀

Example 6.4.2. What sum of money would need to be invested initially to accumulate \$50,000 at a rate of 7.35% from January 31, 2012 to August 18, 2012?

Solution: Here we use equation 6.5 and note that there are 200 days over this period. Students will be given a table to calculate days on quizzes and tests. More about that later.

$$
\begin{aligned}
P &= \frac{S}{1+rt} \\
&= \frac{50000}{1+0.0735\times 200/365} \\
&= 48064.26126\ldots \\
&= \$48064.26
\end{aligned}
$$

So we would need a total of \$48,064.26 under these circumstances. ◀

6.5 Equivalent Values

After working through the first few sections of chapter 6 you may have noticed something very important about money. Its value changes when it is in an investment, bank account, on loan, or used as some other type of financial product. For example, if you borrow money from the bank, the money's value when you first borrow it is less than the value when you pay it back, because you always have to pay back the original loan plus interest.

The value that money changes to after the passage of t years is called a **dated value**, or **equivalent value**. In our examples, this will often times be the same as the future value.

Example 6.5.1. Consider a situation where some company owes a bank for a loan and that loan is due today. However, the company cannot make the payment today and has agreed to pay the loan back in 3 months. Assuming that the bank was charging interest of 6% p.a. on the loan of $40,000. Also assume that because the bank has had such good business with the company, they will not penalize them further but will just charge the added interest. How much will the company have to pay the bank in 3 months?

Solution: Here we use equation 6.4.

$$
\begin{aligned}
S &= P(1 + rt) \\
&= 40000\left(1 + 0.06 \times \frac{3}{12}\right) \\
&= \$40600
\end{aligned}
$$

So our answer is $40600. ◀

Example 6.5.2. Consider another situation where you are expected to pay off a car loan and you are down to your last payment and wish to pay it off 20 days before the end of the month (when it was originally due), how much should you have to pay if the original payment was $714 at 6.75% p.a.?

Solution: Here we use equation 6.5 because in the problem we are given a future value, $714. We are trying to find the value of the money 20 days before it is due.

$$
\begin{aligned}
P &= \frac{S}{1 + rt} \\
&= \frac{714}{1 + 0.0675 \times 20/365} \\
&= 711.368909\ldots \\
&= \$711.37
\end{aligned}
$$

So we would have to pay $711.37. ◀

Example 6.5.3. A debt can be paid off by payments of $872 on year from now and $1180 two years from now. Determine the single payment now that would fully repay the debt. Allow for simple interest at 9%.

Solution: Here we use equation 6.5 twice because in the problem we are given future values, $872 and $1180. We are trying to find the value of the monies 1 year before and 2 years before their respective due dates. Thus we will have to work this problem in two parts and then add the two together. We start with the current value of the \$872 future payment.

$$\begin{aligned} P &= \frac{S}{1+rt} \\ &= \frac{872}{1+0.09(1)} \\ &= \$800 \end{aligned}$$

Now to find the present value of the future \$1180 payment.

$$\begin{aligned} P &= \frac{S}{1+rt} \\ &= \frac{1180}{1+0.09(2)} \\ &= \$1000 \end{aligned}$$

So, to pay off this loan completely today we must send them $800 + 1000 = \$1800$. ◀

This final problem is intended to be somewhat challenging.

Example 6.5.4. You are owed payments of \$400 due today, \$500 due in five months, and \$618 due in one year. You have been approached to accept a single payment nine months from now with interest allowed at 12% p.a. How much will the single payment be?

Solution: In this problem we will have to use a combination of formulae. We will need to use equation 6.4 for the future value of the \$400 and \$500 payments because nine months from now is a future value for them. However, we will need to use equation 6.5 on the \$618 payment because it will be paid off earlier and will this be valued at less than \$618. Again, we deal with these one at a time and sum up our results.

We begin with the future value of the \$400 payment using equation 6.4, $r = 0.12$, and $t = \frac{9}{12}$.

$$\begin{aligned} S &= P(1+rt) \\ &= 400\left(1+0.12\times\frac{9}{12}\right) \\ &= 436 \end{aligned}$$

Now to find the future value of the payment in 5 months of \$500. We have to work carefully here because when this payment is payed off nine months from now it will have accumulated another 4 months of interest. This is what we must calculate. So $P = 500$, $r = 0.12$, and $t = \frac{4}{12}$.

$$\begin{aligned} S &= P(1+rt) \\ &= 500\left(1+0.12\times\frac{4}{12}\right) \\ &= 520 \end{aligned}$$

The final aspect of the problem, the \$618 payment due in one year, is going to be paid off early. Since we are paying it off early we should not have to pay the full \$618, making this a future value in a problem where we need to find the principal, P. So we equation 6.5 with $S = 618$, $r = 0.12$, and $t = \frac{3}{12}$. Note here that if we pay it off nine months from now, that will make this payment exactly 3 months early, thats why we have $t = \frac{3}{12}$.

$$\begin{aligned} P &= \frac{S}{1 + rt} \\ &= \frac{618}{1 + 0.12 \times 3/12} \\ &= 600 \end{aligned}$$

To conclude the problem we just add up these values: $436 + 520 + 600 = \$1556.00$. This is the single payment to pay off all of the debts in the above fashion. ◀

This concludes the chapter on simple interest and the required material for BMAT110-Essentials of Business Mathematics.

Bibliography

[1] S. A. Hummelbrunner, K. Suzanne Coombs, Kelly Halliday, Contemporary Business Mathematics with Canadian Applications, Pearson Canada, Toronto, 9th Edition, 2012.