Supplemental Notes (Incomplete)

Introduction to Calculus

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Contents

1 Review of Limits

To begin we work through a quick review of the concept of limit. This material is strictly for review purposes and is minimal, as the typical student in this course has already taken CALC103. Together with functions, limits are the foundation of calculus and we can do virtually no calculus without them. In this course we do not rigorously prove any of our concepts of limits as this belongs in an advanced calculus or analysis course at university or graduate school of mathematics.

1.1 The Geometry of Limits

Consider the following graph of a function:

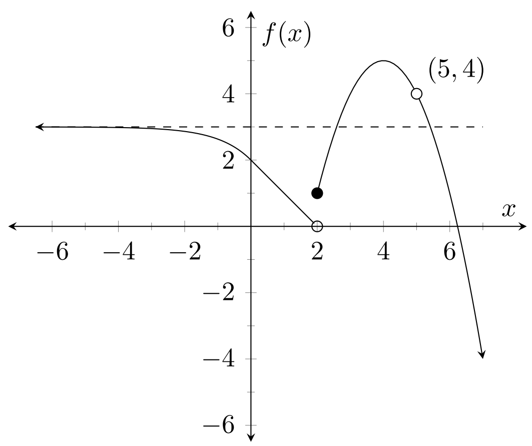


Figure 1.1: Graph of the function $f(x)$

Let us ask the question: What happens to $y = f(x)$ as x gets close to 0 (from either the left hand side or the right hand side of 0?

We can answer this question by simply looking at the picture above and observing what happens to the y values as the x values approach 0. We can see that the graph y value approaches 2. In fact the function is defined to be 2 at $x = 0$. Note that this last point is completely irrelevant. When discussing limits we are not interested in what happens *at* a particular value, but what happens *near* a value.

We now formally restate the question and remind the student how we write the answer to this question in mathematical notation:

Example 1.1.1. Find $\lim_{x \to 0} f(x)$.

Solution: The $\lim_{x \to 0} f(x) = 2$ because as x approaches 0 from both the left and the right hand side, y gets close to 2. ◀

Next, let us consider another problems:

Example 1.1.2. Find $\lim_{x \to 5} f(x)$ (refer to Figure 1.1).

Solution: Again we ask the similar question: What happens to $y = f(x)$ as x gets close to 5? The hole in the graph may confuse some students, however we are reminded that we are not interested in what happens at $x = 5$ only nearby. The open hole means that the function is not even defined at $x = 5$! None-the-less we can look closely and see that the open hole is labeled $(5, 4)$. This tells us that y is getting close to 4 as x approaches 5 from both the left and the right hand side.

Thus $\lim_{x \to 5} f(x) = 4$. ◀

Thirdly, we look at a more curious problem:

Example 1.1.3. Find $\lim_{x \to 2} f(x)$ (refer to Figure 1.1).

Solution: Here the question is: What happens to $y = f(x)$ as x gets close to 2? In this case the gap in the graph may confuse some students, however we are reminded that we are not interested in what happens at $x = 2$ only nearby.

Also note that the limit from the left is different from the limit from the right. We can see that as we approach $x = 2$ from the left hand side of 2, y approaches 0. However, if we let x approach 2 from the right hand side of 2, y approaches 1. So it would seem that the limit from the left is 0 and the limit from the right is 1. We actually have a notation for this in mathematics:

$$\lim_{x \to 2^-} f(x) = 0$$

and

$$\lim_{x \to 2^+} f(x) = 1$$

So it appears that our question has two different answers. Generally, in mathematics our structures cannot have multiple simultaneous values. So in this case we say that $\lim_{x \to 2} f(x)$ is undefined or does not exist. ◀

Property 1.1.1: In general, if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$, then $\lim_{x \to a} f(x) = L$.

and

Property 1.1.2: If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

We have one final question we will ask about Figure 1.1:

Example 1.1.4. Find $\lim_{x \to -\infty} f(x)$.

Solution: In this problem the $-\infty$ symbol refers to large negative numbers. In other words, what happens to $y = f(x)$ as x becomes large negative?

We can see that as the graph of $f(x)$ goes off the left hand side of the graph it tracks along and gets closer to the dashed asymptote at $y = 3$. Thus $\lim_{x \to -\infty} f(x) = 3$. ◀

This concludes our discussion of the geometry of limits.

1.2 Limits of Specific Functions

In the next section we take a look at several examples of very specific functions and ask questions about their limits. In some cases we may not have a graph or picture to look at and considerable work may be required to find some of these limits.

In example 1.1.1 we found the limit of $f(x)$ as x approached 0. The answer was 2, which happened to be the same as $f(0)$. In other words, the value of the limit was the same as the value of the function evaluated at $x = 0$. This happened because there was no "weird" behavior of f "near" $x = 0$. The words "weird" and "near" are used loosely here. We will give this phenomenon a formal name shortly, but let us look at some more examples of where we may take advantage of such lack of "weirdness" to see if we can figure it out.

Example 1.2.1. If $g(x) = x^3 - 1$, find $\lim_{x \to 1} g(x)$.

Solution: Although we could, here we will not use a graph to help us. We will create a table of values to see what happens to $y = g(x)$ as x approaches 1. We need to evaluate $g(x)$ closer and closer to $x = 1$ from both the left and right hand sides.

$x \lesssim 1$	$y = g(x)$	$x \gtrsim 1$	$y = g(x)$
0.8	-0.488	1.2	0.728
0.9	-0.271	1.1	0.331
0.95	-0.142625	1.05	0.157625
0.999	-0.002997	1.001	0.003003

Table 1.1: Evaluation of $g(x)$ Near $x = 1$

We can see from the left two columns of Table 1.1, as we let x get close to 1 from less than 1 ($x \lesssim 1$), $y = g(x) \approx 0$. As we let x approach 1 from the right hand side ($x \gtrsim 1$), $y = g(x) \approx 0$ as well. Thus we conclude that $\lim_{x \to 1} g(x) = 0$. Students should verify the values in the table and further consider checking values of x even closer to 1. ◀

By now many students will see that had we just computed $g(1)$ we would have gotten our limit without all the extra work of constructing Table 1.1. Note that $g(1) = 0$.
The point of Examples 1.1.1 and 1.2.1 is that if there is no "weird" behavior "nearby" then we can find limits by simply substituting the value directly into the function. The

next several definitions and properties will allow us to more precisely understand and discuss this "weirdness' that can occur in functions.

Definition 1.2.1: A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \to c} f(x) = f(c),$$

$y = f(x)$ is **continuous at a left endpoint** a of its domain if

$$\lim_{x \to a^+} f(x) = f(a),$$

$y = f(x)$ is **continuous at a right endpoint** a of its domain if

$$\lim_{x \to a^-} f(x) = f(a),$$

$y = f(x)$ is **continuous** if it is continuous at every point of its domain.
If $y = f(x)$ is is not continuous at a point c, we say that f is **discontinuous** at c and call c a **point of discontinuity**. ◁

These final two properties are key to finding limits of functions easily.

Property 1.2.1: A function $y = f(x)$ is continuous if and only if $\lim_{x \to a} f(x) = f(a)$

Property 1.2.2: Polynomial functions are continuous.

In summary, these properties and definitions tell us that we can find the limit of any polynomial (or any continuous function) by simply evaluating the function at the x value we are interested in approaching. Further, the precise mathematical term for what we are calling "weird" is a point of discontinuity.

Example 1.2.2. If $g(x) = x^3 - 1$, find $\lim_{x \to -3} g(x)$.

Solution: Note here that g(x) is a polynomial function. Thus, according to Properties 1.2.1 and 1.2.2, we may directly substitute $x = -3$ into $g(x)$ to find the limit.

$$\begin{aligned} \lim_{x \to -3} g(x) &= g(-3) \\ &= (-3)^3 - 1 \\ &= -28 \end{aligned}$$

Thus our limit is -28. ◀

Example 1.2.3. If $h(x) = 1 - e^{x+2}$, find $\lim_{x \to 0} h(x)$.

Solution: In this problem we observe that $h(x)$ is not a polynomial function, it is exponential (with $e \approx 2.72$). Exponential functions are continuous as long as their domain is all $\mathbb{R}$. In this case we know that the domain of h is all $\mathbb{R}$ because for any real number x there will be a corresponding range value $h(x)$, thus h is continuous. So by Property 1.2.1, we may directly substitute $x = 0$ into $h(x)$ to find the limit.

$$\begin{aligned}\lim_{x\to 0} h(x) &= h(0)\\ &= 1 - e^{0+2}\\ &= 1 - e^2\\ &\approx -6.38906\ldots\end{aligned}$$

Thus our limit is $1 - e^2$. ◄

One should keep in mind that it is not always possible to find limits quickly because of the possibility of discontinuities. Sometimes our only option is to estimate limits and even then there are situations where estimating limits with tables can fail. Fortunately for us those situations do not often occur in the types of applications we will need this for.

1.3 More on Limits

In this final section on limits we will take a look at some useful properties of limits and look at some fairly common situations where there is an algebraic "trick" to finding a complicated limit quickly. We will call upon the material from this section from time to time to help us prove something or work problems.

Property 1.3.1: Let $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_1$ (where L_1 and L_2 are real numbers,) then:

1. Sum Rule: $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L_1 + L_2$
2. Difference Rule: $\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L_1 - L_2$
3. Product Rule: $\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L_1 \cdot L_2$
4. Constant Multiple Rule: $\lim_{x\to c}[k \cdot f(x)] = k \lim_{x\to c} f(x) = k \cdot L_1$
5. Quotient Rule: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$

Secondly, we take a look at an example of where this is useful:

Example 1.3.1. Find $\lim_{x\to 1} \dfrac{x^3 + 4x^2 - 3}{x^2 + 5}$.

Solution: In this problem note that we are dealing with a rational (fractional) function. Students should be aware that rational functions have domain restrictions such that the denominator cannot be zero. However, in this example the denominator cannot be zero because $x^2 + 5$ is always positive, thus there are no issues going forward.

We begin by using Property 1.3.1 (5) from above:

$$\begin{aligned}\lim_{x\to 1}\frac{x^3+4x^2-3}{x^2+5} &= \frac{\lim\limits_{x\to 1}(x^3+4x^2-3)}{\lim\limits_{x\to 1}(x^2+5)}\\ &= \frac{1^3+4(1)^2-3}{1^2+5}\\ &= \frac{2}{6}\\ &= \frac{1}{3}\end{aligned}$$

Thus $\lim\limits_{x\to 1}\dfrac{x^3+4x^2-3}{x^2+5}=\dfrac{1}{3}$. ◀

In the previous chapter we said that polynomial functions are continuous and that we can find the limits of functions that are continuous by substitution. We now formally state what we previously used.

Theorem 1.3.1: If $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is a polynomial function, and c is any real number, then

$$\lim_{x\to c} f(x)=f(c)=a_nc^n+a_{n-1}c^{n-1}+\cdots+a_1c+a_0$$

◁

The next theorem could have been used to solve the previous example.

Theorem 1.3.2: If $f(x)$ and $g(x)$ are polynomials, and c is any real number, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{f(c)}{g(c)} \qquad \text{provided } g(c)\neq 0.$$

◁

In our next example we will use algebra to help us find the limit of a function at a discontinuity. Note that none of the properties or theorems from above can help use find the limit in the next example.

Example 1.3.2. Find $\lim\limits_{x\to 1}\dfrac{x^2-6x+5}{x-1}$.

Solution: Here we have the limit of a rational function. As we know rational functions are notorious for having discontinuities. This function has a discontinuity at $x=1$ because it is not defined at $x=1$. Thus we need to be a bit more creative or clever here to find the limit. We could construct a table of values however there is a trick that can be used here using elementary algebra.

Note that the numerator factors:

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-6x+5}{x-1} &= \lim_{x\to 1}\frac{(x-5)(x-1)}{x-1}\\ &= \lim_{x\to 1}(x-5)\\ &= 1-5\\ &= -4\end{aligned}$$

The trick is to factor and attempt to reduce the fraction so that the expression in the denominator (that depends on x) will cancel. If the factor in the denominator cancels then this trick will work and you have your limit. If factorization does not cause the denominator to disappear then one has to try something else.

So $\lim_{x\to 1}\frac{x^2-6x+5}{x-1} = -4$. ◀

In the next example we look at another algebraic "trick" that uses a process called rationalizing the numerator.

Example 1.3.3. Find $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$.

Solution: Here we have the limit of another rational function. As we know rational functions are notorious for having discontinuities. This function has a discontinuity at $x = 1$ again because it is not defined at $x = 1$. We could construct a table of values however the trick here is to rationalize the numerator.

Note that:

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} &= \lim_{x\to 1}\left[\frac{x-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\right]\\ &= \lim_{x\to 1}\left[\frac{(x-1)(\sqrt{x}+1)}{x-1}\right]\\ &= \lim_{x\to 1}(\sqrt{x}+1)\\ &= (\sqrt{1}+1)\\ &= 2\end{aligned}$$

So $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} = 2$. ◀

There are many other algebraic "tricks" to finding limits at discontinuities. All that is required to understand them is a little research on the Internet or a textbook. This concludes the section.

1.4 Limits Involving Infinity

In this section we look at limits were x or the domain elements of a function go to $\pm\infty$. We first need to discuss a more precise meaning of these types of limits with a simple example.

Example 1.4.1. Find $\lim_{x\to\infty} \frac{1}{x}$.

Solution: We begin by describing what is meant by $x \to \infty$. This simply means as x gets large positive. Likewise if $x \to -\infty$, then this means x becomes large negative. We can construct a table of values to see what happens to $\frac{1}{x}$ as x gets large positive:

x	$1/x$
1	1
2	0.5
5	0.2
10	0.1
50	0.02
100	0.01
10000	0.0001

Table 1.2: $\frac{1}{x}$ as x gets large

We can see here that $\frac{1}{x}$ is getting close to 0. Thus $\lim_{x\to\infty} \frac{1}{x} = 0$. ◀

A similar argument can be made for the same function as $x \to -\infty$.

Example 1.4.2. Find $\lim_{x\to-\infty} 4$.

Solution: We begin by describing what is meant by $x \to \infty$. This simply means as x gets large positive. Likewise if $x \to -\infty$, then this means x becomes large negative. We can construct a table of values to see what happens to 4 as x gets large positive (table is not really necessary):

x	$f(x)$
-1	4
-2	4
-5	4
-10	4
-50	4
-100	4
-10000	4

Table 1.3: $f(x) = 4$ as x gets large

We can see here that 4 is not changing. Thus $\lim_{x\to\infty} 4 = 4$. Students should have already been aware of the fact that $\lim_{x\to c} k = k$ for any c (including $\pm\infty$) and any k. ◀

A similar set of properties exist for domain values going to infinity as for limits where x goes to a real number.

Property 1.4.1: Let $\lim_{x\to\pm\infty} f(x) = L_1$ and $\lim_{x\to\pm\infty} g(x) = L_1$ (where L_1 and L_2 are real numbers,) then:

1. Sum Rule: $\lim_{x\to\pm\infty}[f(x)+g(x)] = \lim_{x\to\pm\infty} f(x) + \lim_{x\to\pm\infty} g(x) = L_1 + L_2$
2. Difference Rule: $\lim_{x\to\pm\infty}[f(x)-g(x)] = \lim_{x\to\pm\infty} f(x) - \lim_{x\to\pm\infty} g(x) = L_1 - L_2$
3. Product Rule: $\lim_{x\to\pm\infty}[f(x)\cdot g(x)] = \lim_{x\to\pm\infty} f(x) \cdot \lim_{x\to\pm\infty} g(x) = L_1 \cdot L_2$
4. Constant Multiple Rule: $\lim_{x\to\pm\infty}[k\cdot f(x)] = k \lim_{x\to\pm\infty} f(x) = k\cdot L_1$
5. Quotient Rule: $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\pm\infty} f(x)}{\lim_{x\to\pm\infty} g(x)} = \frac{L_1}{L_2}$, if $L_2 \neq 0$

The idea Example 1.4.1 can be expanded for for any function of the form $f(x) = \frac{a}{x^r}$ for some real number a and natural number n.

Theorem 1.4.1: For any fixed real number a and $r \geq 1$, $\lim_{x\to\pm\infty} \frac{a}{x^r} = 0$. ◁

This theorem is easily proven with the properties above.

Finding Limits Involving Infinity

There are more algebraic "tricks" to finding limits involving infinity. Such as when we find the limit of a rational function as $x \to \pm\infty$. As it turns out we can divide all terms of the rational expression by the independent variable raised to the highest power in the denominator then use the facts above to evaluate each limit one at a time.

Example 1.4.3. Find $\lim_{x\to-\infty} \frac{5x^2+8x-3}{3x^2+2}$.

Solution: Here we have the limit of another rational function as x gets large. We cannot simply substitute $-\infty$ in for x because it is not a real number. Perhaps we should look for a way to use the above idea.

Note that:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{5x^2+8x-3}{3x^2+2} &= \lim_{x\to-\infty} \frac{5+8/x-3/x^2}{3+2/x^2} \\
&= \frac{\lim\limits_{x\to-\infty} 5 + \lim\limits_{x\to-\infty} 8/x - \lim\limits_{x\to-\infty} 3/x^2}{\lim\limits_{x\to-\infty} 3 + \lim\limits_{x\to-\infty} 2/x^2} \\
&= \frac{5+0-0}{3+0} \\
&= \frac{5}{3}
\end{aligned}$$

So $\lim\limits_{x\to-\infty} \dfrac{5x^2+8x-3}{3x^2+2} = \dfrac{5}{3}$. ◀

This concludes our discussion of limits involving infinity and our review of limits in this course. Students are encouraged to do their own research to better understand limits and their application as this will better prepare students for this course.

2 Introduction to Differentiation

After developing a clear understand of functions and limits one is typically ready to move on to the two major tiers of calculus, the derivative and integral. Generally the derivative is covered first, but not necessarily always. We will stay with tradition in this course.

In the first section we give a quick reminder of the definition of the derivative of a function. The student should not forget this definition as it useful in generating much of the theory in calculus. In later sections we will review rules of differentiation and work through applications.

2.1 Definition of the Derivative

The process of finding the derivative is called **differentiation**.

Definition 2.1.1: The **derivative** of a function $f(x)$ is the function $\dfrac{df}{dx}$ or $f'(x)$ whose value at x is

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(if the limit exists.)}$$

◁

If the limit in the definition above exists, then we say that f **has a derivative (is differentiable) at** x. If f has a derivative at every point of its domain, we call f **differentiable.**. If f is differentiable, we call its graph a **differentiable curve.**

As you may recall there are many names for the derivative. When $f'(x)$ exists for a particular x, it is called the slope of the curve $y = f(x)$ at x. The line that passes through the point $P(x, f(x))$ with slope $f'(x)$ is the tangent to the curve at P.

There are many common notations for the derivative of a function $y = f(x)$ besides the two above. Some common derivative notation and how we speak it can be found on the next table.

Notation	In English
$\frac{df}{dx}$	"d f d x"
$\frac{d}{dx}(f)$	"d dx of f"
$\frac{dy}{dx}$	"d y d x"
$f'(x)$	"f prime of x"
y'	"y prime"
$y'(x)$	"y prime of x"
$D_x(f)$	"D x of f"

Table 2.1: Common Derivative Notation

It is important to remember that each of these notations have exactly the same meaning.

For the remainder of this section we look at how to use the definition to find the derivative.

Example 2.1.1. Find the derivative of $f(x) = 3x + 2$.

Solution: To find the derivative we must compute the limit from Definition 2.1.1:

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h) + 2 - (3x+2)}{h} \\ &= \lim_{h\to 0} \frac{3x + 3h + 2 - 3x - 2}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3\end{aligned}$$

So the derivative of $f(x)$ is 3. ◀

Example 2.1.2. Find the derivative of $g(x) = -4x^2$.

Solution: Again, to find the derivative we must compute the limit from Definition

2.1.1 using g instead of f:

$$\begin{aligned}\frac{dg}{dx} &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{-4(x+h)^2+4x^2}{h}\\ &= \lim_{h\to 0}\frac{-4(x^2+2xh+h^2)+4x^2}{h}\\ &= \lim_{h\to 0}\frac{-4x^2-8xh-h^2+4x^2}{h}\\ &= \lim_{h\to 0}\frac{-8xh+h^2}{h}\\ &= \lim_{h\to 0}\frac{h(-8x+h)}{h}\\ &= \lim_{h\to 0}(-8x+h)\\ &= -8x\end{aligned}$$

So the derivative of $g(x)$ is $-8x$. ◀

Unfortunately, as we deal with increasingly more complicated functions the definition becomes impractical for finding the derivative. Consider the function $h(x) = 5x^{200} - 3x^{150} + 30x^{50}$, which is very tedious to differentiate with the definition. In the next section we look at shortcut methods, one of which will make h easy to differentiate.

2.2 Rules for Differentiation

Here we look at a few basic rules of differentiation.

Differentiation of a Power Function

We begin with likely the most important differentiation rule.

Definition 2.2.1: A **Power Function** is any function of the form

$$f(x) = x^n,$$

with any real number $n \neq 0$. ◁

As you may have seen in a previous course the derivative of a power function is given quickly with the power rule.

Proposition 2.2.1: If n is a fixed positive integer and $f(x) = x^n$, then

$$\frac{df}{dx} = nx^{n-1}.$$

Proof. In order to prove this proposition we need the Binomial Theorem from elementary algebra:

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-1}h^2 + \ldots + h^n$$

The proof uses the definition of the derivative and the above formula with $a = x$ and $b = h$:

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-1}h^2 + \ldots + h^n\right) - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-1}h^2 + \ldots + h^n}{h}
\end{aligned}$$

The important thing to realize at this point is that each of the terms in the numerator have a multiple of h. Thus we can factor out h.

$$\begin{aligned}
&= \lim_{h\to 0} \frac{h\left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}h + \ldots + h^{n-1}\right)}{h} \\
&= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}h + \ldots + h^{n-1}\right)
\end{aligned}$$

At this point we are dealing with a polynomial in terms of the variable x because n is constant. Thus we can let h go to zero. Notice all of the terms except for the first become zero.

$$\begin{aligned}
&= nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}(0) + \ldots + (0)^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

Thus $\frac{df}{dx} = nx^{n-1}$, the desired result. □

The Power Rule can be further generalized for any function of the form $f(x) = ax^n$ where a is any real number. Limit Laws 4 helps us here:

$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}(ax^n) \\
&= a\frac{d}{dx}(x^n) \\
&= a\left(nx^{n-1}\right) \\
&= anx^{n-1}
\end{aligned}$$

We illustrate this and the Power Rule in the next example.

Example 2.2.1. Find the derivative of $y = -7x^5$.

Solution: Using the power rule we have

$$\begin{aligned}\frac{dy}{dx} &= -7(5)x^{5-1} \\ &= -35x^4\end{aligned}$$

So the derivative of y is $-35x^4$. ◀

We can also extend the Power Rule to sums and differences of Power Functions.

Theorem 2.2.1: If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

◁

We omit the proof of this theorem, as it is quite simple.

Example 2.2.2. Write the equation of the line that is tangent to $f(x) = 1 - 2x - 3x^2$ and passes through $(-2, -7)$.

Solution: We need to things to write the equation of the line: the slope and a point. Conveniently we have the point, we just need to find the slope of the tangent line (the derivative) of y at $x = -2$ (the x value of the point).

$$\frac{df}{dx} = -2 - 6x$$

We now institute some notation. The following implies that we are to evaluate the derivative at $x = -2$.

$$\begin{aligned}\left.\frac{df}{dx}\right|_{x=-2} &= -2 - 6(-2) \\ &= 10 && \text{The slope of our line}\end{aligned}$$

Next we use the slope and the point to write the equation of our line. Here we use slope-intercept form with m as the slope and b as the y-intercept.

$$\begin{aligned}y &= mx + b \\ -7 &= 10(-2) + b \\ 13 &= b\end{aligned}$$

Thus our line has slope 10 and y-intercept 13, giving us the equation of our line as $y = 10x + 13$. ◀

Finally, you can see a graph of the previous problem.

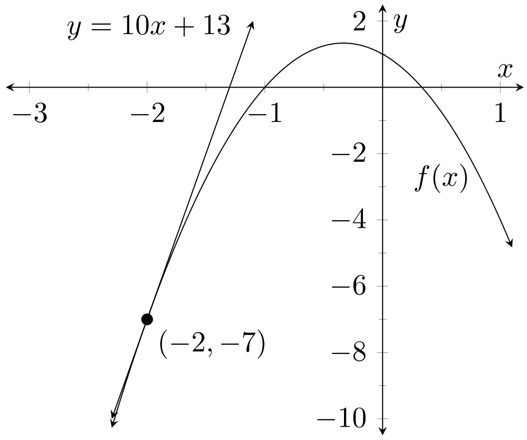


Figure 2.1: Graph of $f(x)$ and its Tangent Line

This concludes our section on the basic rules of differentiation.

2.3 Differentiation of Composite Functions

In this section we take a look at a very powerful differentiation rule, the Chain Rule. In concert with the Power Rule, the Chain Rule will allow us to differentiate a large collection of Real Functions.

Definition 2.3.1: Let $y(x)$ and $u(x)$ be real valued functions where the domain of y is the range of u, then any function of the form

$$y(u(x))$$

is called a **composite function**. ◁

Example 2.3.1. The following functions are composite:

1. $f_1(x) = (x+4)^3$ is composite where $f_1(x) = y(u(x))$ with $y(u) = u^3$ and $u(x) = x + 4$.

2. $f_2(x) = \sqrt{x^2 - 1}$ is composite where $f_2(x) = y(u(x))$ with $y(u) = \sqrt{u}$ and $u(x) = x^2 - 1$.

3. $f_3(x) = e^{\sin x}$ is composite where $f_3(x) = y(u(x))$ with $y(u) = e^u$ and $u(x) = \sin x$.

To simplify our notation we will simply refer to $y(u)$ as y and $u(x)$ as u from this point forward.

Naturally, we would like to know how to differentiate such functions since many functions can be expressed as composite. The **Chain Rule** will enable us to do so.

Proposition 2.3.1: (**Chain Rule**) Let f, y, and u be differentiable functions with $domain(y) = range(u)$ such that $f(x) = y(u(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Proof. We actual detailed proof is somewhat more complicated, however, for our purposes a more compact version will be enough.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dx} \cdot 1 \\ &= \frac{dy}{dx} \cdot \frac{du}{du} \\ &= \frac{dy}{du} \cdot \frac{du}{dx} \end{aligned}$$

This proof depends on each of the differentials (dy, du, and dx) being small, but not zero. It should also be noted that many of the conclusions in calculus depend on the idea that differentials are small but not zero. □

The main application of the Chain Rule is to find the decomposition of a function (if it can be expressed as composite) then use the chain rule to find its derivative.

Example 2.3.2. Find the derivative of $f(x) = (x+4)^3$ without expanding.

Solution: We first need to find the decomposition of $f = y(u)$. Here $y(u) = u^3$ and $u(x) = x + 4$. Next, we apply the Chain Rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 3u^2 \cdot 1 \\ &= 3(x+4)^2 \end{aligned}$$

Thus $\frac{dy}{dx} = 3(x+4)^2$. Also, it is important not to forget to remove the dummy variable u. Many students may forget and leave u in their answer. This is a common mistake. ◀

Example 2.3.3. Find the derivative of $y = \frac{5}{(x^2-1)^2}$.

Solution: First, we rewrite y with properties of exponents as $y = 5(x^2-1)^{-2}$. Secondly, we need to find the decomposition of $f = y(u)$. Here $y(u) = 5u^{-2}$ and $u(x) = x^2 - 1$. Next, we apply the Chain Rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -10u^{-3} \cdot 2x \\ &= -20x(x^2-1)^{-3} \end{aligned}$$

Thus $\frac{dy}{dx} = -20x(x^2-1)^{-3}$. In some circumstances it may be more useful to write this as $\frac{dy}{dx} = \frac{-20x}{(x^2-1)^3}$. ◀

We work one last problem.

Example 2.3.4. Find the slope of the tangent line at $x = 1$ on the function $h(x) = 2(x^4 - 2x^2)^5$.

Solution: First, we need to find the decomposition of $h = y(u)$. Here $y(u) = 2u^5$ and $u(x) = x^4 - 2x^2$. Next, we apply the Chain Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 10u^4 \cdot (4x^3 - 4x) \\ &= 10(x^4 - 2x^2)^4 \cdot (4x^3 - 4x) \\ &= 10(4x^3 - 4x)(x^4 - 2x^2)^4\end{aligned}$$

Thus $\frac{dy}{dx} = 10(4x^3 - 4x)(x^4 - 2x^2)^4$. Next we need to evaluate the derivative at $x = 1$:

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=1} &= 10\left[4(1)^3 - 4(1)\right]\left[(1)^4 - 2(1)^2\right]^4 \\ &= 10(-1)^4(4-4) \\ &= 0\end{aligned}$$

So the slope of the tangent line is zero, concluding our problem. ◀

As we conclude this section students should keep in mind that the Chain Rule is a very useful tool throughout calculus. There are many situations where it can be used in some form or fashion that ends by making complex problems easier.

2.4 Differentiation of Product and Quotient Functions

In this section we are interested in differentiating functions that can be written as products and quotients of other functions. We first motivate differentiation of a function made of the product of two functions.

Proposition 2.4.1: (**Product Rule**) Let u and v be differentiable functions such that $y(x) = u(x) \cdot v(x)$, then

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Proof. We begin by assuming that $y(x) = u(x) \cdot v(x)$, then we compute its derivative directly using the definition.

$$
\begin{aligned}
\frac{dy}{dx} = \frac{d(uv)}{dx} &= \lim_{h\to 0} \frac{y(x+h)-y(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)\cdot v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)\cdot v(x+h) - v(x+h)u(x) + v(x+h)u(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{v(x+h)[u(x+h)-u(x)] + u(x)[v(x+h)-v(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{v(x+h)[u(x+h)-u(x)]}{h} + \frac{u(x)[v(x+h)-v(x)]}{h}\right] \\
&= \lim_{h\to 0} \frac{v(x+h)[u(x+h)-u(x)]}{h} + \lim_{h\to 0} \frac{u(x)[v(x+h)-v(x)]}{h} \\
&= v(x) \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} + u(x) \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \\
&= v\frac{du}{dx} + u\frac{dv}{dx}
\end{aligned}
$$

as desired. □

Example 2.4.1. Differentiate $y = 3x^4\sqrt{2-3x}$.

Solution: We proceed with the Product Rule with $u(x) = 3x^4$ and $v(x) = \sqrt{2-3x}$.

$$
\begin{aligned}
\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\
&= \sqrt{2-3x}(12x^3) + 3x^4\left(\frac{1}{2}(2-3x)^{-\frac{1}{2}}(-3)\right) \\
&= 12x^3\sqrt{2-3x} - \frac{9}{2}x^4(2-3x)^{-\frac{1}{2}} \\
&= 12x^3\sqrt{2-3x} - \frac{9x^4}{2\sqrt{2-3x}}
\end{aligned}
$$

Note that the final answer could take on several different forms. ◀

There is a similar formula for differentiating function that can be written as the quotient of two functions. For example: $f(x) = \dfrac{u(x)}{v(x)}$ or $\dfrac{u}{v}$.

Proposition 2.4.2: (**Quotient Rule**) Let u and v be differentiable functions such that $y(x) = \dfrac{u(x)}{v(x)}$ with $v(x) \neq 0$, then

$$
\frac{dy}{dx} = \frac{d(u/v)}{dx} = \frac{vu' - uv'}{v^2}
$$

using prime notation.

The proof of the Quotient Rule is very similar to the Product Rule, thus we omit it here.

Example 2.4.2. Take the derivative of $y = \dfrac{2x^3}{4x+1}$.

Solution: We proceed with the Quotient Rule with $u(x) = 2x^3$ and $v(x) = 4x + 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2}\\ &= \frac{(4x+1)(6x^2) - 2x^3(4)}{(4x+1)^2}\\ &= \frac{6x^2(4x+1) - 8x^3}{(4x+1)^2}\\ &= \frac{24x^3 + 6x^2 - 8x^3}{(4x+1)^2}\\ &= \frac{16x^3 + 6x^2}{(4x+1)^2}\end{aligned}$$

Note again that the final answer could take on several useful forms. ◀

Students are cautioned that with the Product Rule the order makes no difference: $uv' + vu' = vu' + uv'$, however, with the Quotient Rule the order of the terms in the numerator is significant because subtraction is not commutative.

2.5 Implicit Differentiation

Up to now, the variable in the function has been the same variable that we take the derivative with respect to, as in the following examples (in terms of the Chain Rule):

Example 2.5.1. We find each derivative and use the Chain Rule even though it is not needed here:

1. $\dfrac{d}{dx}(x^3) = 3x^2\dfrac{dx}{dx} = 3x^2$
2. $\dfrac{d}{dt}(t^4) = 4t^3\dfrac{dt}{dt} = 4t^3$

What if we are differentiating a function that depends on one variable with respect to another variable? Like the next example:

Example 2.5.2. Find $\dfrac{d}{dx}(2t^3)$. Assume that t depends on x.

Solution: Note here that we are differentiating with respect to x, but the function depends on t, which in turn depends on x. To work this we must apply the Chain Rule:

$$\frac{d}{dx}(2t^3) = 6t^2\frac{dt}{dx}$$

Students may find the form of our answer disturbing, but we can go no further because we do not know the exact relationship between t and x. ◀

Example 2.5.3. Find the derivative $\frac{d}{dt}(2y - t^3)$. Assume that y depends on t.

Solution: Here note that we are taking the derivative with respect to t and that y depends on t.

$$\frac{d}{dt}(2y - t^3) = 2\frac{dy}{dt} - 3t^2$$

Concluding our problem. ◀

Normally we think of y as the dependent variable but in the next problem we think of x as the dependent variable.

Example 2.5.4. Find the derivative $\frac{dx}{dy}$, for $x = y^2 - 3y + 2$.

Solution: In this problem we are taking the derivative with respect to y and note here that x depends on y.

$$\frac{dx}{dy} = 2y - 3$$

Concluding our problem. ◀

We now turn our attention to **Implicit Functions** or functions in which the dependent variable is not isolated on one side of the equation. In many cases Implicit Functions have dependent variables that cannot be isolated in any way. Conveniently, there is a method in calculus for differentiating such functions. We now discuss this highly general process and note that we will make use of the concepts above.

Example 2.5.5. Differentiate $x^2 + y^2 = y^3 - x$ with respect to x. Assume that y depends on x.

Solution: In this problem we are taking the derivative with respect to x and note that y depends on x.

$$2x + 2y\frac{dy}{dx} = 3y^2\frac{dy}{dx} - 1$$

Next, just solve the equation for $\frac{dy}{dx}$ and we are done.

$$\begin{aligned} 2y\frac{dy}{dx} - 3y^2\frac{dy}{dx} &= -2x - 1 \\ (2y - 3y^2)\frac{dy}{dx} &= -2x - 1 \\ \frac{dy}{dx} &= \frac{2x + 1}{3y^2 - 2y} \end{aligned}$$

Concluding our problem. ◀

Example 2.5.6. Find the slope of the tangent line at $x = 0$ on the ellipse

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{9}\right)^2 = 1$$

Solution: We begin by rewriting our implicit function:

$$\begin{aligned} \frac{x^2}{16} + \frac{y^2}{81} &= 1 \\ 81x^2 + 16y^2 &= 1296 \qquad \text{Multiply both sides by 1296} \end{aligned}$$

In this problem we are taking the derivative with respect to x and assume that y depends on x.

$$\begin{aligned} 162x + 32y\frac{dy}{dx} &= 0 \\ 32y\frac{dy}{dx} &= -162x \\ \frac{dy}{dx} &= \frac{-162x}{32y} \\ \frac{dy}{dx} &= -\frac{81x}{16y} \end{aligned}$$

As we can see from the derivative above we need an x value and a y value. So we use one of the two original equations to find the y value.

$$\begin{aligned} \frac{x^2}{16} + \frac{y^2}{81} &= 1 \\ \frac{(0)^2}{16} + \frac{y^2}{81} &= 1 \\ \frac{y^2}{81} &= 1 \\ y^2 &= 81 \end{aligned}$$

Which tells us that there are two y values for $x = 0$: $y = 9$ and $y = -9$. So now we can go back to our derivative formula and substitute. First, we use $x = 0$ and $x = 9$:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{81x}{16y} \\ &= -\frac{81(0)}{16(-9)} \\ &= -\frac{0}{-144} \\ &= 0 \end{aligned}$$

Secondly, we use $x = 0$ and $y = -9$:

$$\begin{aligned}\frac{dy}{dx} &= -\frac{81x}{16y}\\ &= -\frac{81(0)}{16(9)}\\ &= -\frac{0}{144}\\ &= 0\end{aligned}$$

So there are two tangents lines at $(0, 9)$ and $(0, -9)$, each with slope zero. See the graph below for a look at the graph and the tangent lines. The tangent lines are the dashed lines, which clearly have a slope of zero. ◀

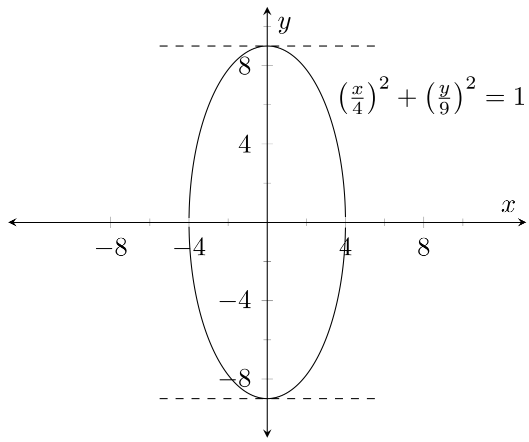


Figure 2.2: Illustration of Previous Problem

Differentials

In the final part of this section we discuss differentials. Informally, a differential, dy, is a very small change in y. So dx is a very small change in x. More formally,

Definition 2.5.1: Let y be a differentiable function of x, then

$$dy = f'(x)dx$$

is called the **differential of** y spoken "d y". ◁

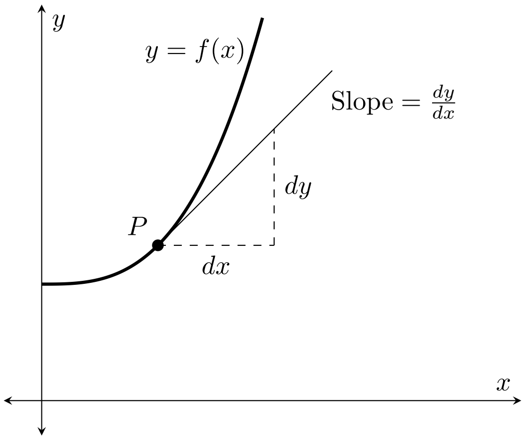


Figure 2.3: Differentials dy and dx

For purposes that will become apparent later in the course we need to be able to identify dy. The process is generally, quite simple. We treat dx as if it were a variable and solve the equation for dy. One can also just use the definition to write dy.

Example 2.5.7. Find the differential dy, if $y = 6x^4 - 2x^2 - 1$.

Solution: We begin by finding $\frac{dy}{dx}$, then to finish, solve for dy:

$$\frac{dy}{dx} = 24x^3 - 4x$$
$$dy = (24x^3 - 4x)dx$$

Concluding our problem. ◀

Example 2.5.8. Find the differential dy of the implicit function $x^2 + y^2 = y^3 - x$. Assume y depends on x.

Solution: Note that we have found this implicit derivative earlier:

$$\frac{dy}{dx} = \frac{2x+1}{3y^2 - 2y}$$
$$dy = \left(\frac{2x+1}{3y^2 - 2y}\right) dx$$

Concluding our problem. ◀

This brings us to the end of other section.

2.6 Higher-Order Differentiation

It certainly makes sense that we could differentiate iteratively. In other words, take the derivative of the derivative or we could take the derivative of the derivative of the derivate (as many times as we like). The derivative of the derivative is called the second derivative. The derivative of the derivative of the derivative or the derivative of the second derivative is called the third derivative. Our notation for such higher-order derivatives follows:

$$\frac{d^2y}{dx^2} \text{ or } y'' \text{ or } f''(x) \text{ or } D^2y$$

We will use many of these notations as practice throughout this section.

Example 2.6.1. Find the first four derivatives of $f(x) = 3x^5 + 2x^4 - 5x^3 - x^2 + 7$.

Solution: We list the first four derivatives by differentiating f four times and recording the result at each step:

$$\begin{aligned}
\frac{df}{dx} &= 15x^4 + 8x^3 - 15x^2 - 2x \\
\frac{d^2f}{dx^2} &= 60x^3 + 24x^2 - 30x - 2 \\
\frac{d^3f}{dx^3} &= 180x^2 + 48x - 30 \\
\frac{d^4f}{dx^4} &= 360x + 48
\end{aligned}$$

Students should be clear that each of the derivatives above is our answer in this case, as the problem asks for each of the first four derivatives. ◄

Example 2.6.2. Find the third derivatives of $G(r) = \sqrt{r} + \sqrt[3]{r}$. Leave your answer as a rational exponent.

Solution: First, we rewrite $G(r)$ in terms of rational exponents so that we can use the power rule.

$$G(r) = r^{\frac{1}{2}} + r^{\frac{1}{3}}$$

Next, we find the first and second derivatives so that we can find the third:

$$\begin{aligned}
G'(r) &= \frac{1}{2}r^{-\frac{1}{2}} + \frac{1}{3}r^{-\frac{2}{3}} \\
G''(r) &= -\frac{1}{4}r^{-\frac{3}{2}} - \frac{2}{9}r^{-\frac{5}{3}} \\
G^{(3)}(r) &= \frac{3}{8}r^{-\frac{5}{2}} + \frac{10}{27}r^{-\frac{8}{3}}
\end{aligned}$$

Thus the third derivative of $G(r)$ is $\frac{3}{8}r^{-\frac{5}{2}} + \frac{10}{27}r^{-\frac{8}{3}}$. ◄

This brings our section on Higher-Order Derivatives to a close and our chapter on an introduction to Differentiation.

3 Differentiation of the Transcendental Functions

In general it is not so easy to differentiate transcendental functions, however in this section we will focus on some of the "elementary" transcendental functions: Trigonometric, Logarithmic, and Exponential Functions.

3.1 Differentiation of Sine and Cosine Functions

As we begin to discuss the calculus of the Trigonometric Functions we must recall something related to them. In particular, there are many methods of measuring angles, but for the purpose of calculus we will use radian measure only from this point on. Thus the argument of a Trigonometric Function will always be in units of radians. We will not review these functions further as students should have encountered these functions in a previous course.

First, we state a theorem and two lemmas and will prove only the lemmas. These are necessary to derive the derivative of the Sine Function.

Theorem 3.1.1: (**The Sandwich Theorem**) Let I be an interval having the point a as a limit point. Let f, g, and h be functions on I, except possibly at a and suppose that for every x in I not equal to a, we have:

$$g(x) \leq f(x) \leq h(x)$$

and also suppose that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then

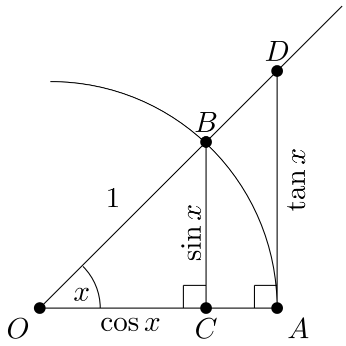
$$\lim_{x \to a} f(x) = L$$

◁

We do not prove the Sandwich Theorem because the required theory is beyond the scope of this course.

Lemma 3.1.1: If x is a small angle (measured in radians), then $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

Proof. In proving the Lemma above we will make use of the diagram below.



Let x be a small angle, in radians, in a circle of radius 1 (see figure above), in which

$$\sin x = \frac{BC}{1} = BC$$
$$\cos x = \frac{OC}{1} = OC$$
$$\tan x = \frac{AD}{1} = AD$$

It is easy to see that the area of the sector OAB is greater than the area of triangle OBC but less than the area of triangle OAD. Now noting that

$$\text{area of triangle } OBC = \frac{1}{2}BC \cdot OC = \frac{1}{2}\sin x \cos x$$

and

$$\text{area of triangle } OAD = \frac{1}{2}OA \cdot AD = \frac{1}{2}\tan x$$

both, by the area of a triangle formula.

Further, by the formula for finding the area of a sector of a circle:

$$\text{area of sector } OAB = \frac{1}{2}r^2 x = \frac{x}{2}$$

So we now have that

$$\frac{1}{2}\sin x \cos x < \frac{x}{2} < \frac{1}{2}\tan x$$

which by multiplying everything by $\frac{2}{\sin x}$ we arrive at

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$

If we now let everything go to zero we know that $\lim_{x \to 0}(\cos x) = 1$ and $\lim_{x \to 0}\left(\frac{1}{\cos x}\right) = 1$ and thus by the Sandwich Theorem $\lim_{x \to 0}\left(\frac{x}{\sin x}\right) = 1$ and finally by reciprocating we can conclude that $\lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 1$, proving the lemma. □

We can also see this informally by looking at the graph of $f(x) = \dfrac{\sin x}{x}$:

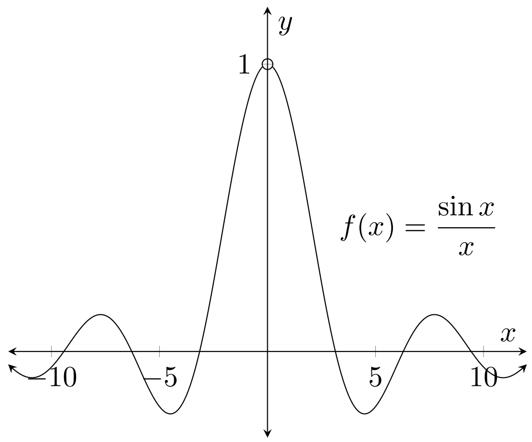


Figure 3.1: Graph of $f(x) = \dfrac{\sin x}{x}$

Even though the function f is not defined at zero we can see that both the left and right hand limits approach one.

Lemma 3.1.2: If x is a small angle (measured in radians), then $\lim_{x\to 0} \dfrac{\cos(x) - 1}{x} = 0$.

Proof. We begin by noting that

$$(1 - \cos x)(1 + \cos x) = \sin^2 x$$

so

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x) - 1}{x} &= \lim_{x\to 0} \frac{-\sin^2 x}{x(1 + \cos x)} \\
&= \lim_{x\to 0} \frac{-(\sin x)(\sin x)/x}{1 + \cos x} \\
&= \frac{-0(1)}{2} && \text{Direct Substitution and Lemma 3.1.1} \\
&= 0
\end{aligned}$$

Concluding our proof. □

We now state and prove the derivative of the Sine Function.

Proposition 3.1.1: Let x be an angle measured in radians and $f(x) = \sin x$, then $\dfrac{df}{dx} = \cos x$.

Proof. We use the definition of the derivative here.

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{[\sin x\cos h+\cos x\sin h]-\sin x}{h} && \text{Sum angle formula} \\
&= \lim_{h\to 0}\frac{\cos x\sin h-\sin x+\sin x\cos h}{h} \\
&= \lim_{h\to 0}\frac{\cos x\sin h-\sin x(1-\cos h)}{h} \\
&= \lim_{h\to 0}\frac{\cos x\sin h}{h}-\lim_{h\to 0}\frac{\sin x(1-\cos h)}{h} \\
&= (\cos x)\lim_{h\to 0}\frac{\sin h}{h}-(\sin x)\lim_{h\to 0}\frac{(1-\cos h)}{h} \\
&= (\cos x)(1)-(\sin x)(0) && \text{Lemmas 3.1.1 and 3.1.2} \\
&= \cos x
\end{aligned}$$

giving us the desired conclusion. □

Now that we have seen considerable theory and derived the derivative of the Sine Function. We will point out that the derivative of the Cosine Function is developed in a similar fashion and simply give the derivative.

Proposition 3.1.2: Let x be an angle measured in radians and $f(x)=\cos x$, then $\frac{df}{dx}=-\sin x$.

We now move to working a few examples.

Example 3.1.1. Differentiate $y=3x^2-\cos(4x)$.

Solution: We use the previous rules in concert with the proposition immediately above to differentiate.

$$\begin{aligned}
\frac{dy}{dx} &= 6x+\sin(4x)\times 4 \\
&= 6x+4\sin(4x)
\end{aligned}$$

giving us our solution. ◀

Example 3.1.2. Differentiate $y=-\sin(6x^3)\cos(2x)$. Do not simplify beyond cleaning up signs.

Solution: We use the previous rules in concert with the propositions above to differentiate.

$$\begin{aligned}
\frac{dy}{dx} &= \cos(2x)(-\cos(6x^3)(18x^2)+\sin(6x^3)\sin(2x)(2) \\
&= -18x^2\cos(2x)\cos(6x^3)+2\sin(6x^3)\sin(2x)
\end{aligned}$$

giving us the solution. ◀

This concludes our introductory section on Differentiation of the Trigonometric Functions.

3.2 Differentiation of the Tangent, Cotangent, Secant, and Cosecant Functions

In this section we complete the differentiation formulas for the Trigonometric Functions. We will derive the derivative of the Tangent Function and leave the rest for students to derive.

Proposition 3.2.1: Let x be an angle measured in radians and $f(x) = \tan x$, then $\dfrac{df}{dx} = \sec^2 x$.

Proof. To find the derivative of the Tangent Function we begin by noting that

$$\tan x = \frac{\sin x}{\cos x}$$

then using the Quotient Rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos^2 x + \sin^2 x}{\cos^x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

which concludes our proof. □

The rest of the differentiation formulas are listed in the table below.

$\dfrac{d}{dx}[\sec x] = \sec x \tan x$	$\dfrac{d}{dx}[\csc x] = -\csc x \cot x$	$\dfrac{d}{dx}[\cot x] = -\csc^2 x$

Example 3.2.1. Differentiate $y = 3\tan(x^2)$. Do not simplify beyond cleaning up signs.

Solution: Using the formulas and note we need the Chain Rule here we have

$$\begin{aligned}\frac{dy}{dx} &= 3\sec^2(x^2)(2x) \\ &= 6x\sec^2(x^2)\end{aligned}$$

giving us the solution. ◀

Example 3.2.2. Differentiate $y = 4\sec^3(x)$. Do not simplify beyond cleaning up signs.

Solution: First we should note that $\sec^3 x = (\sec x)^3$ thus we need the Chain Rule.

$$\begin{aligned}\frac{dy}{dx} &= 4 \times 3(\sec x)^2 \times \sec x \tan x \\ &= 12 \tan x \sec^3 x\end{aligned}$$

giving us the derivative. ◀

Example 3.2.3. Using implicit differentiation, find the derivative, $\frac{dy}{dx}$ where $xy + y \cot x = 0$.

Solution: Here we work very carefully using various rules and using implicit differentiation:

$$\begin{aligned}1y + x\frac{dy}{dx} + \frac{dy}{dx}\cot x - y\csc^2 x = 0 & \qquad \text{Chain Rule} \\ x\frac{dy}{dx} + \frac{dy}{dx}\cot x = y\csc^2(x) - y & \\ \frac{dy}{dx}(x + \cot x) = y\csc^2(x) - y & \\ \frac{dy}{dx} = \frac{y\csc^2(x) - y}{x + \cot x} &\end{aligned}$$

giving us the derivative. ◀

This concludes our section and brings us to the end of our formal discussion of derivatives of the Trigonometric Functions.

3.3 Differentiation of Exponential and Logarithmic Functions

The course textbook will break this section into separate sections, however, in the notes we will treat Exponential and Logarithmic Functions as they naturally exist, as two sides of the same mathematical object. It should therefore come as no surprise that their derivatives are intertwined as well. We will begin by finding the derivative of a special exponential function, then use this to find the derivative of Logarithmic Functions and from there find the derivative of all Exponential Functions.

The Special Exponential Function

We begin this section by reminding ourselves about just exactly what an exponential function is.

Definition 3.3.1: Let a be a positive real number and n be any real number, then any function of the form

$$f(x) = a^x$$

is called a **Simple Exponential Function**. ◁

We could certainly create more complex exponential functions by adding, subtracting, multiplying, and dividing with other functions. We could also increase the complexity by taking roots or further exponentiating the simple exponential function. None-the-less once we learn to differentiate $f(x) = a^x$ we will be able to differentiate another more complex exponential function via our previous rules for differentiating sums, differences, products, quotients, and composite functions.

Of course this being a calculus course we will naturally ask the question: How do we take the derivative of $f(x) = a^x$? To answer this question we resort back to our definition of the derivative.

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{a^x(x+h)-a^x}{h}\\ &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\ &= a^x \lim_{h\to 0}\frac{a^h-1}{h}\end{aligned}$$

Note that this last expression implies that the derivative of a^x depends on itself, a^x. We should also be aware at this point that we have not generated a simple formula for the derivative of $f(x) = a^x$.

We now propose that there is a special choice of a in $f(x) = a^x$ such that $\lim_{h\to 0}\frac{a^h-1}{h} = 1$. If we can find that special choice of a, then we have found a special function whose derivative is the same as the function itself!

Taking a look at the graph below we have used a computer to plot a graph of the limit above.

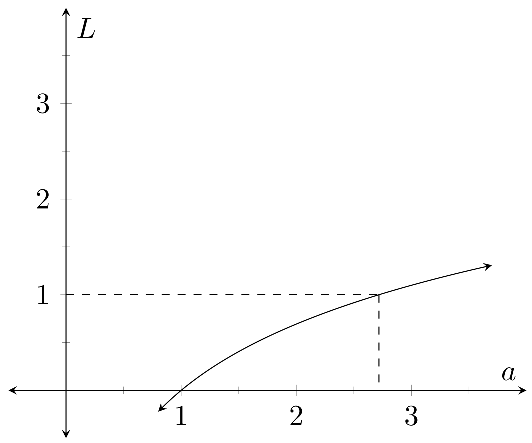


Figure 3.2: Graph of $L = \lim_{h\to 0}\frac{a^h-1}{h}$

Simulation and the graph above should verify to the student that this special choice is approximately $a \approx 2.718$. You will recognize this number from a previous course. It is sometimes called Euler's Number.

Definition 3.3.2: Euler's Number, denoted $e \approx 2.718$, is an irrational number that when substituted in for a, solves the equation $\lim_{h \to 0} \frac{a^h - 1}{h} = 1$ ◁

Thus, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. The student should verify this limit equation is true by creating a table of values with h approaching zero.

In conclusion to all of the above we have a new differentiation formula:

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Thus the function $f(x) = e^x$ is the function who is its own derivative. Believe it or not this simple differentiation formula will allow us to differentiate a huge collection of transcendental functions.

Differentiation of Logarithmic Functions

We now move on to Logarithmic Functions. Logarithmic Functions are the inverses of Exponential Functions.

Definition 3.3.3: Let a be a positive real number, then $f(x) = \log_a x$ is called the base a **Logarithmic Function**. ◁

An example of a logarithmic function is $f(x) = \log_e x$ which is sometimes called the **Natural Logarithmic Function** because it has Euler's Number as its base. We usually write it $y = \ln x$. It is the inverse function of $y = e^x$ discussed earlier. This means that $y = \log_e x$ is the same as $x = e^y$. As a result of this fact we can find the derivative of $f(x) = y$ using implicit differentiation.

$$\begin{aligned} x &= e^y \\ 1 &= e^y \frac{dy}{dx} && \text{Differentiating both sides} \\ \frac{1}{e^y} &= \frac{dy}{dx} \\ \frac{1}{x} &= \frac{dy}{dx} \end{aligned}$$

Thus

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

We can further differentiate all logarithmic functions with any base a using the same idea.

Proposition 3.3.1: The derivative of $y = \log_a x$ is $y' = \dfrac{1}{x \ln a}$.

Proof. Note here that $y = \log_a x \Leftrightarrow x = a^y$. Thus

$$\begin{aligned}
x &= a^y \\
1 &= e^{\ln a^y} && \text{Law of Logarithms} \\
1 &= e^{y \ln a} && \text{Another Law of Log's} \\
1 &= e^{y \ln a} \cdot \ln a \frac{dy}{dx} \\
\frac{1}{e^{y \ln a} \cdot \ln a} &= \frac{dy}{dx} \\
\frac{1}{e^{a^y \cdot \ln a}} &= \frac{dy}{dx} \\
\frac{1}{x \ln a} &= \frac{dy}{dx}
\end{aligned}$$

giving us the desired conclusion. □

Thus

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

Differentiation of Exponential Functions

As our final bit of theory for this section we derive a formula for differentiating the simple exponential function for any base a. Thus we want to differentiate $y = a^x$.

We begin by noting that by the definition of logarithms $y = a^x \Leftrightarrow x = \log_a y$. We will use the right hand side and differentiate implicitly.

$$\begin{aligned}
x &= \log_a y \\
1 &= \frac{1}{y \ln a} \frac{dy}{dx} && \text{Proposition above} \\
y \ln a &= \frac{dy}{dx} && \text{Multiply both sides by } y \ln a \\
a^x \ln a &= \frac{dy}{dx} && \text{Substitute } y \text{ from above}
\end{aligned}$$

So

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

We now have a complete list of differentiation formulas for the elementary transcendental functions. The rest of the section will be spent working through a few examples.

Example 3.3.1. Differentiate $y = \log_3(2x^3 + 2x)$.

Solution: We differentiate here using proposition 3.3.1 and the chain rule.

$$\begin{aligned} y' &= \frac{1}{(2x^3+2x)\ln 3}(6x^2+2) \\ &= \frac{6x^2+2}{(2x^3+2x)\ln 3} \end{aligned}$$

concluding our problems. ◀

Example 3.3.2. Differentiate $y = e^x \cdot 3^{3x+1}$.

Solution: We differentiate here using the product and the chain rules. We let $u = e^x$ and $v = 3^{3x+1}$

$$\begin{aligned} y' &= u'v + uv' \\ &= e^x 3^{3x+1} + 3e^x \cdot 3^{3x+1}\ln 3 \\ &= (1+3\ln 3)e^x \cdot 3^{3x+1} && \text{Factor} \end{aligned}$$

is our derivative. Either of the last two lines would be an acceptable expression for the derivative. ◀

Example 3.3.3. Differentiate $f(x) = \ln(\ln x)$.

Solution: We differentiate here using the chain rules.

$$\begin{aligned} f'(x) &= \frac{1}{\ln x}\cdot\frac{1}{\ln x} \\ &= \frac{1}{(\ln x)^2} \end{aligned}$$

concluding our problems. ◀

We now come to the end of the formal discussion of differentiation.

4 Introduction to Integration

In this chapter we begin discussion of the other major tier of calculus, integration. If you treat Differentiation and Integration as functions we can easily show that they are inverses.

4.1 The Indefinite Integral

To understand the Integral we must first discuss the rather intuitive idea of an *antiderivative*.

Definition 4.1.1: An **antiderivative** of an expression is a new expression which, if differentiated, gives the original expression. ◁

After reading this definition one should be thinking that the derivative and antiderivative are *inverse operations*, for lack of a better word. In other words, the derivative of the antiderivative of a function is the function.

Example 4.1.1. The derivative of x^4 is $4x^3$, thus an antiderivative of $4x^3$ is x^4.

Referring to the example above we should also not that there are infinitely many antiderivatives of any expression or function.

Example 4.1.2. The derivative of x^4+3 is $4x^3$, thus an antiderivative of $4x^3$ is x^4+3, because the derivative of a constant is zero.

In general, the antiderivative of $4x^3$ is x^4+c where c is any constant real number.

Let us be a bit more rigorous now. If $F(x)+c$ is a function, then its derivative is

$$\frac{d}{dx}\left[F(x)+c\right] = F'(x)+0 = F'(x)$$

Further, and because dx is very small (but not zero) we can multiple both sides by dx and convert the above into differential form which gives us

$$d\left[F(x)+c\right] = F'(x)dx$$

Thus the differential of $F(x)+c$ is $F'(x)dx$. Conversely, the antiderivative of $F'(x)dx$ is $F(x)+c$. Mathematicians have shortened this notation somewhat by replacing the phrase "antiderivative of" with the integral sign $\int$. So

Definition 4.1.2: Let $F(x)$ be a differentiable function, then the **Indefinite Integral** of F' with respect to x is

$$\int F'(x)\,dx = F(x)+c$$

The expression or function that we are finding the antiderivative of is called the **Integrand**. In this case the integrand is $F'(x)$. ◁

Example 4.1.3. Find the indefinite integral of $g(x) = 2x$.

Solution: To solve this problem we need to think of a function whose derivative is $2x$. We should know that as x^2. Thus

$$\int 2x\,dx = x^2 + c$$

So $x^2 + c$ is the general antiderivative of $2x$. ◀

Before we move on we should say that it is customary to replace $F'(x)$ with $f(x)$ in the definition of the integral. Thus $F'(x) = f(x)$ which allows us to rewrite the definition as follows

Definition 4.1.3: Let $F(x)$ be a differentiable function whose derivative is $f(x)$, then the **Indefinite Integral** of f with respect to x is

$$\int f(x)\,dx = F(x) + c$$

◁

From this point on we will use this convention.

We should also mention that the constant c is usually refereed to as the **constant of integration**.

Some Integration Rules

Here we introduce some of the very basic rules and properties of integrals.

Property 4.1.1:

$$\int dx = x + c$$

The next few properties are some fairly standard properties that should be no surprise about integrals.

Property 4.1.2:

$$\int af(x)\,dx = a\int f(x)\,dx + aF(x) + c$$

Example 4.1.4. [**?**, p.884]Find $\displaystyle\int 3kx^2\,dx$, if k is a constant.

Solution: We already know that the antiderivative of $3x^2$ is x^3. We simply apply the property above to pull the k out of the picture.

$$\begin{aligned}\int 3kx^2\,dx &= k\int 3x^2\,dx\\ &= kx^3 + c\end{aligned}$$

Concluding our problem. ◀

Example 4.1.5. Find $\int 7\,dx$.

Solution: Again 7 is a constant so pull it out in front of the integral sign.

$$\begin{aligned}\int 7\,dx &= 7\int dx\\ &= 7x + c\end{aligned}$$

Concluding our problem. ◀

The following property is also true for differences.

Property 4.1.3:

$$\int [f_1(x) + f_2(x) + \cdots]\ dx = \int f_1(x)\,dx + \int f_2(x)\,dx + \cdots + c$$

The next property is just the power rule from our discussion of differentiation in reverse.

Property 4.1.4: Let n be any real number except for -1, then

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

Example 4.1.6. Find $\int \frac{1}{3}x^5\,dx$.

Solution: Here we use a combination of properties.

$$\begin{aligned}\int \frac{1}{3}x^5\,dx &= \frac{1}{3}\int x^5\,dx\\ &= \frac{1}{3}\frac{x^6}{6} + c\\ &= \frac{1}{18}x^6 + c\\ &= \frac{x^6}{18} + c\end{aligned}$$

which is our answer. ◀

Example 4.1.7. [**?**, p.887] Find $\int \frac{x^5-2x^3+5x}{x}\,dx$.

Solution: Here again, we use a combination of properties.

$$\begin{aligned}\int \frac{x^5-2x^3+5x}{x}\,dx &= \int \left(x^4 - 2x^2 + 5\right)\,dx\\ &= \int x^4 - 2\int x^2\,dx + 5\int dx\\ &= \frac{x^5}{5} - \frac{2x^3}{3} + 5x + c\end{aligned}$$

which is our solution. ◀

Example 4.1.8. [**?**, p.887] Find $\int \frac{x^3 - x^2 + 5x - 5}{x - 1}\, dx$.

Solution: In this problem we can use long division to simplify the quotient, then integrate.

$$\begin{array}{r} x^2 + 5 \\ x - 1 \overline{) x^3 - x^2 + 5x - 5} \\ \underline{x^3 - x^2} \\ 0 + 5x - 5 \\ \underline{5x - 5} \end{array}$$

Thus $\frac{x^3-x^2+5x-5}{x-1} = x^2 + 5$, which can easily be integrated.

$$\begin{aligned} \int \frac{x^3 - x^2 + 5x - 5}{x - 1}\, dx &= \int x^2 + 5\, dx \\ &= \frac{x^3}{3} + 5x + c \end{aligned}$$

which is our solution. ◀

This concludes our introductory section on integration.

4.2 Basic Integration Rules

In this section we add some of the basic integration methods derived from differentiation. First, we will take a look at reversing the chain rule. With respect to integration this is referred to as the **Substitution Rule** or **u-Substitution**.

We know from the chain rule that if f, y, and u are differentiable functions such that $f(x) = y(u(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We learned this in a previous chapter. If we write the above in prime notation and convert to differential form we have

$$dy = y'(u) \cdot u'(x)\, dx$$

and now integrate both sides

$$\int y'(u) \cdot u'(x)\, dx = y(u(x)) + c$$

The above derivation is the Substitution Method. In less mathematical terms, it says that, we can integrate a function if the integrand contains:.

- a composite function of the form $y(u(x))$,

- the derivative of $u(x)$, $u'(x)$.

In using the Substitution Method students are highly encouraged to clearly state u, $\dfrac{du}{dx}$ and to convert $\dfrac{du}{dx}$ to differential form.

Example 4.2.1. Find $\displaystyle\int 2x\sqrt{1+x^2}\,dx$.

Solution: In this example note that the integral is not straightforward. But notice that there is a composite function:

$$y(u) = \sqrt{u} \qquad\qquad \text{where } u = 1 + x^2$$

and further,

$$du = 2x\,dx$$

It should be clear after looking at the above, for a moment, that our integrand is of the form $y'(u) \cdot u'(x)$. So by substituting from above we have

$$\begin{aligned}\int 2x\sqrt{1+x^2}\,dx &= \int \sqrt{u}\,du \\ &= \int u^{\frac{1}{2}}\,du \\ &= \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c \\ &= \frac{2}{3}\sqrt{(1+x^2)^3} + c\end{aligned}$$

which is our answer. Students should verify that this is correct by differentiating. Do not forget to replace u each time you use the Substitution Method. ◀

Example 4.2.2. Find $\displaystyle\int \sqrt{2x+1}\,dx$.

Solution: In this example note that the integral is not straightforward. But notice that there is a composite function:

$$y(u) = \sqrt{u} \qquad\qquad \text{where } u = 2x + 1$$

and further,

$$du = 2\,dx \text{ and } \frac{du}{2} = dx$$

In this cause we do not see 2 in the integrand, but let us see what we can do

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int \frac{\sqrt{u}}{2}\,du\\ &= \frac{1}{2}\int u^{\frac{1}{2}}\,du\\ &= \frac{1}{3}u^{\frac{3}{2}}+c\\ &= \frac{1}{3}(2x+1)^{\frac{3}{2}}+c\\ &= \frac{1}{3}\sqrt{(2x+1)^3}+c\end{aligned}$$

which is our answer. Again, students should verify that this is correct by differentiating and don't forget to replace u. ◀

Example 4.2.3. Find $\int x^3\cos(x^4+2)\,dx$.

Solution: Notice again that there is a composite function:

$$y(u) = \cos u \qquad\qquad \text{where } u = x^4+2$$

and further,

$$du = 4x^3\,dx \text{ and } \frac{du}{4} = x^3\,dx$$

and so

$$\begin{aligned}\int x^3\cos(x^4+2)\,dx &= \int \frac{\cos u}{4}\,du\\ &= \frac{1}{4}\int \cos u\,du\\ &= \frac{1}{4}\sin(u)+c\\ &= \frac{1}{4}\sin(x^4+2)+c\end{aligned}$$

which is our answer. ◀

At this point students should see that the Substitution Method can be employed as long as the derivative of u with respect to the variable and exponent are the same. Coefficients do not have to match, as they can be adjusted by solving for dx.

Miscellaneous Integration Rules Derived from Differentiation

Next, we generate a small list of integration rules that come from reversing rules that we have used while taking the derivative.

Property 4.2.1: If $x \neq 0$, then

$$\int \frac{dx}{x} = \ln|x| + c$$

Property 4.2.2: If x is an angle in radians, then

$$\int \cos x \, dx = \sin(x) + c$$

Property 4.2.3: If x is an angle in radians, then

$$\int -\sin x \, dx = \cos(x) + c$$

Property 4.2.4: If x is an angle in radians, then

$$\int \sec^2 x \, dx = \tan(x) + c$$

Property 4.2.5: If x is a real number, then

$$\int e^x \, dx = e^x + c$$

Of course this list can be extended for other derivatives that we know or have a formula of. We work one more example for this section.

Example 4.2.4. Find $\displaystyle\int \frac{4z}{3z^2 - 5} \, dz$.

Solution: In this case it may be difficult to see but there is a composite functions whose derivative (with respect to variable and exponent) exists within the integrand.

$$y(u) = \frac{1}{u} \qquad \text{where } u = 3z^2 - 5$$

and further,

$$du = 6z \, dz \text{ and } \frac{du}{6} = z \, dz$$

and so

$$\begin{aligned}\int \frac{4z}{3z^2 - 5} \, dz &= \int \frac{4}{6} \frac{1}{u} \, du \\ &= \frac{2}{3} \int \frac{1}{u} \, du \\ &= \frac{2}{3} \ln|u| + c \\ &= \frac{2}{3} \ln|3z^2 - 5| + c\end{aligned}$$

which is our integral. ◀

This brings us to the end of another section.

4.3 Constant of Integration

Here we will be taking a quick look at applications of integration that will allow us to identify specific functions from a given derivative and other information. It is not difficult to see that in the previous sections we have been looking at entire families of functions as a result of integrating. The family of functions results from the lack of clarity about the constant c.

In this section we will be dealing what are called **Differential Equations**. These equations come up in many applications, across many fields of inquiry.

Definition 4.3.1: A **Differential Equation** is an equation that contains a derivative. ◁

Example 4.3.1. Graph the function whose derivative is $\frac{dy}{dx} = 2x$.

Solution: As we know there are infinitely many functions whose derivative is $2x$. We can find the family of functions by taking the integral of the given differential equation.

$$\begin{aligned} \frac{dy}{dx} &= 2x \\ dy &= 2x\,dx && \text{Separate Variables} \\ \int dy &= \int 2x\,dx \\ y &= x^2 + c \end{aligned}$$

Note that the second line above is called **Separation of Variables**, this will come up later.

We will now graph y above for a few integer values of c, but note that c can be any real number.

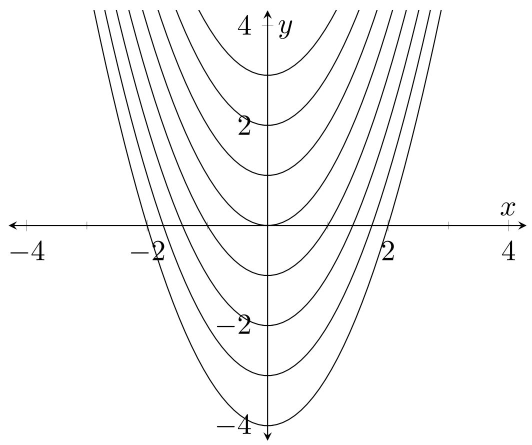


Figure 4.1: Graph of $y = x^2 + c$ family of functions

This concludes our problem. ◀

In application, we are often interested in a specific function, not a family. In order to specify a particular function we need more than just the derivative to find c, we need a **boundary condition** or **initial condition**. The phrase initial condition is usually (but not always) reserved for when the independent variables is time.

Example 4.3.2. Find the function whose derivative is $\frac{dy}{dx} = 2x$ and passes through the point $(2, 9)$.

Solution: We have already found the family of functions in the previous problem.

$$y = x^2 + c$$

Next, we just need to substitute the boundary condition into the function.

$$\begin{aligned} y &= x^2 + c \\ 9 &= 2^2 + c \\ 5 &= c \end{aligned}$$

Thus our desired function is $y = x^2 + 5$. ◀

Example 4.3.3. Find the function whose second derivative is $\frac{d^2y}{dx^2} = 4$ and passes through the point $(2, 6)$ and is parallel to $y = 3x$.

Solution: Here we will need to integrate twice.

$$\begin{aligned} y'' &= 4 \\ y' &= \int 4\,dx \\ &= 4x + c_1 \end{aligned}$$

At this point we can use the fact that the slope (first derivative) of the function we are looking for is 3 because the slope of $y = 3x$ is 3 at the point we are interested in.

$$\begin{aligned} 3 &= 4(2) + c_1 \\ -5 &= c_1 \end{aligned}$$

Thus we have more information about our function: its derivative function is $y = 4x - 5$. Now we can integrate again.

$$\begin{aligned} y' &= 4x - 5 \\ y &= \int 4x - 5\,dx \\ &= 2x^2 - 5x + c_2 \end{aligned}$$

Finally, we can use the boundary condition to find c_2.

$$\begin{aligned} y &= 2x^2 - 5x + c_2 \\ 6 &= 2(2)^2 - 5(2) + c_2 \\ 6 &= -2 + c_2 \\ 8 &= c_2 \end{aligned}$$

So our final answer is the function $y = 2x^2 - 5x + 8$ whose second derivative is 4 and satisfies the conditions above. ◀

This brings our section to a close.

4.4 The Definite Integral

In this section we define what is called the definite integral from the indefinite integral.

In the previous section we discussed finding the indefinite integral. Here we expand on this in terms of an integral function. Consider

$$F(x) = \int f(x)\,dx$$

We could evaluate this function just like any previously studied function. We would first have to find the indefinite integral and then substitute in whatever x is required. Let us look at a specific example.

Example 4.4.1. Compute $F(1) = \int 2x\,dx$.

Solution: To find $F(1)$ we must first find the indefinite integral.

$$\begin{aligned} F(x) &= \int 2x\,dx \\ &= x^2 + c \end{aligned}$$

So $F(x) = x^2 + c$, which can now be evaluated.

$$\begin{aligned} F(1) &= (1)^2 + c \\ &= 1 + c \end{aligned}$$

We can do nothing else until we know more about c under these circumstances. ◀

Now that we understand that we can evaluate an integral function at a particular x, we can talk about adding, subtracting, multiplying, or dividing such functions easily.

The Definite Integral

Let us consider evaluating the integral function $F(x) = \int f(x)\,dx$ at $x = a$ and $x = b$. This would give us $F(a)$ and $F(b)$. Further the result in each case would be some number plus c. If we then subtracted, note that the two resulting constants would cancel away leaving just the difference of two numbers. This is the definite integral.

Definition 4.4.1: Let f be an integrable function on the interval $[a, b]$ with $a \leq b$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is called the **Definite Integral** from a to b. ◁

The rest of this section is spent practicing the definite integral.

Example 4.4.2. Compute $\int_0^2 x^3\,dx$.

Solution: To begin we use the definition of the definite integral and find the indefinite integral of x^3.

$$\begin{aligned}\int_0^2 x^3\,dx &= F(2) - F(0)\\ &= \frac{1}{4}(2)^4 - \frac{1}{4}(0)^4\\ &= 4 - 0\\ &= 4\end{aligned}$$

Thus $\int_0^2 x^3\,dx = 4$. ◀

Example 4.4.3. Compute $\int_{-1}^3 (3x-2)^4\,dx$.

Solution: In this case we may want to find the indefinite integral of $(3x-2)^4$ first. We can use u-substitution, with $u = 3x - 2$ and $du = 3\,dx$ which is $\frac{du}{3} = dx$.

$$\begin{aligned}\int (3x-2)^4\,dx &= \int \frac{u^4}{3}\,du\\ &= \frac{1}{3}\int u^4\,du\\ &= \frac{1}{3}\left(\frac{u^5}{5} + c\right)\\ &= \frac{(3x-2)^5}{15} + c\end{aligned}$$

Note that once we introduce the definite integral, the constant c will disappear. Further,

$$\begin{aligned}\int_{-1}^3 (3x-2)^4\,dx &= \frac{(3(3)-2)^5}{15} - \frac{(3(-1)-2)^5}{15}\\ &= \frac{7^5}{15} - \frac{(-5)^5}{15}\\ &= \frac{6644}{5}\end{aligned}$$

Thus $\int_{-1}^3 (3x-2)^4\,dx = \frac{6644}{5}$. ◀

Example 4.4.4. Compute $\int_0^1 ze^{(z^2)}\,dz$.

Solution: Again, we begin by finding the indefinite integral of ze^{z^2} with respect to z. We can use u-substitution, with $u = z^2$ and $du = 2z\,dz$ which is $\frac{du}{2} = z\,dz$. We will work slowly and carefully here.

$$\begin{aligned}
\int ze^{(z^2)}\,dz &= \int e^{(x^2)}(z\,dz) \\
&= \int \frac{e^u}{2}\,du \\
&= \frac{1}{2}\int e^u\,du \\
&= \frac{1}{2}\left(e^u + c\right) \\
&= \frac{e^u}{2} + c \\
&= \frac{e^{(z^2)}}{2} + c
\end{aligned}$$

Note that once we introduce the definite integral, the constant c will disappear. Further,

$$\begin{aligned}
\int_0^1 ze^{(z^2)}\,dz &= \frac{e^{(1^2)}}{2} - \frac{e^{(0^2)}}{2} \\
&= \frac{e}{2} - \frac{1}{2} \\
&= \frac{e-1}{2}
\end{aligned}$$

Thus $\int ze^{(z^2)}\,dz = \frac{e-1}{2}$. ◀

This concludes our section on the definite integral.

4.5 Area Under the Curve of a Function

In a previous course you would have learn the theory about how the definite integral corresponds to the area under the curve from $x = a$ to $x = b$. Here we again formalize this and practice a few problems.

Theorem 4.5.1: Let f be an integrable function on the interval $[a, b]$ with $a \leq b$, then the exact area under the curve of f is given by the definite integral

$$A = \int_a^b f(x)\,dx$$

where the area of the region is further bound by the x axis, and the lines $x = a$ and $x = b$. ◁

We illustrate this with the next diagram.

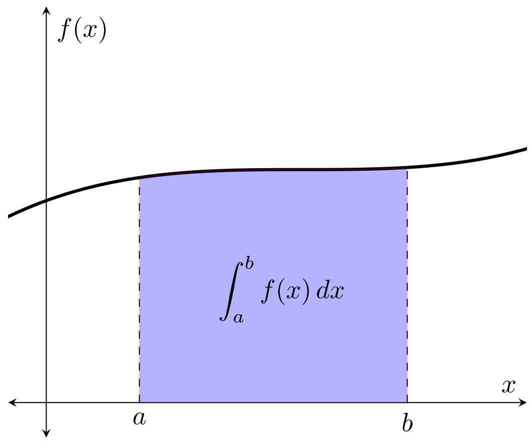


Figure 4.2: Area under the Curve of $f(x)$

Looking at the final problem in the last section by rephrasing:

Example 4.5.1. Find the area between the curve $f(x) = (3x-2)^4$, the x axis, and the vertical lines $x = -1$ and $x = 3$.

Solution: To solve this problem we must compute:

$$\int_{-1}^{3} (3x-2)^4\,dx$$

We found in the previous section that this definite integral evaluates to $\frac{6644}{5}$. Thus

$$\text{Area under the curve} = \int_{-1}^{3} (3x-2)^4\,dx = \frac{6644}{5} = 1328.8 \text{ units}^2$$

Concluding the problem. ◀

Example 4.5.2. Find the area between the curve $y = ze^{(z^2)}$, the x axis, and the vertical lines $z = 0$ and $z = 1$.

Solution: To solve this problem we must compute:

$$\int_{0}^{1} ze^{(z^2)}\,dz$$

We found in the previous section that this definite integral evaluates to $\frac{e-1}{2}$. Thus

$$\text{Area under the curve} = \int_{0}^{1} ze^{(z^2)}\,dz = \frac{e-1}{2} \approx 0.859 \text{ units}^2$$

Concluding the problem. ◀

Example 4.5.3. Find the area under the curve of the function $g(x) = \sin x$ from $x = 0$ to $x = \pi$.

Solution: To solve this problem we must compute:

$$\int_0^\pi \sin x \, dx$$

To begin we must find the indefinite integral corresponding the above definite integral.

$$\int \sin x \, dx = -\cos(x) + c$$

Next evaluate the definite integral

$$\begin{aligned}\int_0^\pi \sin x \, dx &= -\cos(\pi) - (-\cos 0) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Thus

$$\text{Area under the curve} = \int_0^\pi \sin x \, dx = 2 \text{ units}^2$$

Concluding the problem. ◀

5 Integration Methods

In this chapter we focus on various integration methods. We begin with some simple integration formulas for Transcendental Functions and then move on to methods that are somewhat more tedious than the methods previously discussed in the course.

5.1 Integrals of Exponential and Logarithmic Functions

Here we look at methods for integrating Exponential and Logarithmic Functions. We already know the integral for $f(x) = e^x$.

$$\int e^x \, dx = e^x + c$$

Further, we can use u Substitution to make the formula above more versatile.

Here we are more interested in integrating a more general class of Exponential Function. Consider the function $y = a^x$ and the question of what is its integral. We begin by rewriting a^x with properties of exponents and use u substitution.

$$\begin{aligned}\int a^x \, dx &= \int e^{\ln a^x} \, dx \\ &= \int e^{x \ln a} \, dx\end{aligned}$$

Next, let $u = x \ln a$ and $du = \ln a \, dx$, further $\dfrac{du}{\ln a} = dx$. So continuing from above

$$\begin{aligned}\int \frac{e^u}{\ln a} \, du &= \frac{1}{\ln a} \int e^u \, du \\ &= \frac{1}{\ln a} \left(e^u + c\right) \\ &= \frac{e^u}{\ln a} + c \\ &= \frac{e^{\ln a^x}}{\ln a} + c \\ &= \frac{a^x}{\ln a} + c\end{aligned}$$

So

Property 5.1.1: Let a be an positive real number, then

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

Example 5.1.1. Find $\displaystyle\int 18x^2\left(4^{2x^3}\right)\,dx$.

Solution: In this example we use substitution and the property above with $u = 2x^3$, $du = 6x^2\,dx$, and $\dfrac{du}{6} = x^2\,dx$. So

$$\begin{aligned}\int 18x^2\left(4^{2x^3}\right)\,dx &= 18\int\left(4^{2x^3}\right)x^2\,dx\\ &= 18\int \frac{4^u}{6}\,du\\ &= \frac{18}{6}\int 4^u\,du\\ &= 3\left(\frac{4^u}{\ln 4} + c\right)\\ &= \frac{3\cdot 4^u}{\ln 4} + c\end{aligned}$$

concluding our problem. ◀

Next, we move to integrating logarithms. We give the next formula without proof for the time being. The proof will be given in a coming section.

Property 5.1.2: Let x be a positive real number, then

$$\int \ln x\,dx = x\ln(x) - x + c$$

Example 5.1.2. Find $\displaystyle\int x\ln\left(3x^2\right)\,dx$.

Solution: In this example we use substitution again and the property above with $u = 3x^2$, $du = 6x\,dx$, and $\dfrac{du}{6} = x\,dx$. So

$$\begin{aligned}\int x\ln\left(3x^2\right)\,dx &= \int \ln\left(3x^2\right)x\,dx\\ &= \frac{1}{6}\int \ln u\,du\\ &= \frac{1}{6}\left(u\ln(u) - u + c\right)\\ &= \frac{3x^2\ln(3x^2) - 3x^2}{6} + c\\ &= \frac{x^2\ln(3x^2) - x^2}{2} + c\end{aligned}$$

concluding our problem. ◀

Finally, we use the previously unproven property to see how to integrate a general logarithmic expression. We use a property of logarithms that you should be familiar with from a previous algebra class. It is called the Change of Base Formula.

Property 5.1.3: Let a and x be positive real numbers, then

$$\int \log_a x\,dx = \frac{x\ln(x) - x}{\ln a} + c$$

Proof. We begin by assuming that a and x are positive real numbers, then

$$\begin{aligned}
\int \log_a x\,dx &= \int \frac{\ln x}{\ln a}\,dx && \text{Change of Base} \\
&= \frac{1}{\ln a}\int \ln x\,dx && \text{Integral of } \ln x \\
&= \frac{x\ln(x) - x}{\ln a} + c
\end{aligned}$$

as desired. □

Example 5.1.3. Find $\int \log(3x-7)\,dx$.

Solution: Again we use substitution and the property above with $u = 3x - 7$, $du = 3\,dx$, and $\frac{du}{3} = dx$. So

$$\begin{aligned}
\int \log(3x-7)\,dx &= \int \frac{\log u}{3}\,du \\
&= \frac{1}{3}\int \log u\,du \\
&= \frac{1}{3}\left(\frac{u\ln(u) - u}{\ln 10} + c\right) \\
&= \frac{(3x-7)\ln(3x-7) - 3x + 7}{3\ln 10} + c
\end{aligned}$$

We can leave the answer in this form. ◀

This concludes our section.

5.2 Trigonometric Integrals

Here we look at integration of the remainder of the Transcendental Functions for this course. Some of these integrals that are trivial to find with the Substitution Method, others will require theory from a later section. We present them here with proof for the first three.

Property 5.2.1: Let x be some real number, then

$$\int \sin x\,dx = -\cos(x) + c$$

Proof. Begin with the assumptions in the property statement, we rewrite the integral as

$$\begin{aligned}\int \sin x\, dx &= -\int -\sin x\, dx \\ &= -\cos(x) + c\end{aligned}$$

Concluding our proof. □

Property 5.2.2: Let x be some real number, then

$$\int \cos x\, dx = \sin(x) + c$$

Proof. The integral should be obvious here because $\frac{d}{dx}(\sin(x) + c) = \cos x$. □

Property 5.2.3: Let x be some real number, then

$$\int \tan x\, dx = -\ln|\cos x| + c$$

Proof. To prove this we use a trigonometric identity and substitution with $u = \cos x$, $du = -\sin x\, dx$, and $-du = \sin x\, dx$.

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\ &= \int \frac{1}{u}\, du \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + c\end{aligned}$$

Concluding our proof. □

Property 5.2.4: Let x be some real number, then

$$\int \cot x\, dx = \ln|\sin x| + c$$

Property 5.2.5: Let x be some real number, then

$$\int \sec x\, dx = \ln|\sec x + \tan x| + c$$

Property 5.2.6: Let x be some real number, then

$$\int \csc x\, dx = \ln|\csc x + \cot x| + c$$

All of the rules above are applied in similar fashion. In many cases student will need to apply u substitution.

Example 5.2.1. Find $\int_0^1 x \sin x^2 \, dx$.

Solution: Here we have a definite integral. We must begin by finding the indefinite integral. We can us Substitution with $u = x^2$, $du = 2x\,dx$, and $\frac{du}{2} = x\,dx$. So we have

$$\begin{aligned}
\int_0^1 x \sin x^2 \, dx &= \int_{u_0}^{u_1} \frac{\sin u}{2} \, du \\
&= \frac{1}{2} \int_{u_0}^{u_1} \sin u \, du \\
&= -\frac{1}{2} \left[\cos u\right]_{u_0}^{u_1} \\
&= -\frac{1}{2} \left[\cos x^2\right]_0^1 \\
&= -\frac{1}{2} \left[\cos 1 - \cos 0\right] \\
&\approx 0.2298
\end{aligned}$$

This concludes our problem. ◀

We come to the end of another section.