

Supplemental Notes

# Mathematics of Finance

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# 1 Compound Interest

In this chapter we talk about compound interest and briefly discuss applications. We begin by contrasting simple and compound interest. Simple interest is the amount of money charged for money borrowed. With simple interest the charged amount of money is calculated only once. Compound interest is essentially simple interest where interest is periodically added back to the principal. Thus the amount of interest, with compound interest, could be calculated many times, unlike simple interest. In the table below we illustrate the difference between simple and compound interest.

Year	At Compound Interest		At Simple Interest	
	Interest Computation	Amount	Interest Computation	Amount
	Original principal	\$10,000.00		\$10,000.00
1	Add interest (0.10)(10,000.00)	1000.00	(0.10)(10,000.00)	1,000.00
	Amount end Year 1	11,000.00		11,000.00
2	Add interest (0.10)(11,000.00)	1100.00	(0.10)(10,000.00)	1,000.00
	Amount end Year 2	12,100.00		12,000.00
3	Add interest (0.10)(12,100.00)	1210.00	(0.10)(10,000.00)	1,000.00
	Amount end Year 3	13,310.00		13,000.00
4	Add interest (0.10)(13,310.00)	1331.00	(0.10)(10,000.00)	1,000.00
	Amount end Year 4	14,641.00		14,000.00
5	Add interest (0.10)(14,641.00)	1464.10	(0.10)(10,000.00)	1,000.00
	Amount end Year 5	16,105.10		15,000.00
6	Add interest (0.10)(16,105.10)	1610.51	(0.10)(10,000.00)	1,000.00
	Amount end Year 6	17,715.61		16,000.00

Table 1.1: Compound Interest Versus Simple Interest for a Principal of \$10,000 Invested at 10% per Annum for 6 Years

## 1.1 The Compound Interest Formula

It should be noted that this section corresponds to section 9.1 in the course textbook.

### 1.1.1 Basic Construction

The question that arises from the information above is how do we convert it into a more usable and manageable form? One possible answer to this question is to use mathematics

by converting everything into symbols and replicating the arithmetic above with algebra.

We begin by recalling the simple interest formula from previous mathematics courses

$$S = P + Pin$$

where  $P$  is the initial amount of money placed into a financial product (i.e. savings account),  $i$  is the periodic rate of interest,  $S$  is the value of the investment after  $n$  periods, and  $n$  is the number of periods over the term of the financial product.

We now use simple interest in the construction of compound interest, where the interest is added back to the initial investment,  $P$ , at the end of each period as in Table 1.1. So the value of the investment at the end of the first compounding period will be

$$\begin{aligned} S_1 &= P + Pi(1) && \text{From simple interest formula} \\ &= P(1 + i) && \end{aligned} \tag{1.1}$$

Now to calculate the value of the investment after the second compounding period we have  $S_1$  as the new initial amount, giving us

$$\begin{aligned} S_2 &= S_1 + S_1i(1) \\ &= P(1 + i) + P(1 + i)i && \text{From Equation 1.1 above} \\ &= P(1 + i)(1 + i) && \text{Factor out } P(1 + i) \\ &= P(1 + i)^2 && \end{aligned} \tag{1.2}$$

Let us calculate one more. To find the value of the investment after the third compounded period we just use  $S_2$  as the new principal

$$\begin{aligned} S_3 &= S_2 + S_2i(1) \\ &= P(1 + i)^2 + P(1 + i)^2i && \text{From Equation 1.2 above} \\ &= P(1 + i)^2(1 + i) && \text{Factor out } P(1 + i)^2 \\ &= P(1 + i)^3 \end{aligned}$$

At this point a pattern should be observed. In particular, the value of the investment after  $n$  compounding periods is

$$S_n = P(1 + i)^n \tag{1.3}$$

Equation 1.3 is one of the more widely used forms of the compounding interest formula in the business community. In fact we will most likely replace  $S_n$  with just  $S$  from this point on. Another form of Equation 1.3 is

$$FV = PV(1 + i)^n \tag{1.4}$$

Here  $FV$  should not be confused as  $F \times V$  and  $PV$  should not be confused with  $P \times V$ . In this context they simply mean future value and present value respectively. Equations 1.3 or 1.4 may be used interchangeably. To avoid confusion we will use Equation 1.3 from this point forward unless requested otherwise.

It is worthwhile to mention that we will refer to the factor  $(1 + i)^n$  as the **compounding factor** or **accumulation factor** later on.

### 1.1.2 Nominal (annual) rate of interest

We discussed previously the periodic rate of interest and represented it with the variable  $i$ . As it turns out, in application,  $i$  is rarely given. Often we are given the annual or nominal rate of interest. The periodic rate of interest is found by dividing the nominal rate of interest by the number of compounding periods per year. So if  $r$  is the nominal rate of interest and  $m$  is the number of compounding periods per year then the periodic rate of interest is

$$i = \frac{r}{m} \quad (1.5)$$

In the financial community compounding frequency,  $m$ , is expressed as with the language in the table below.

Compounding Frequency	Length of Compounding Period	Number of Compounding Periods per Year, $m$
Annual	1 year (12 months)	1
Semi-annual	6 months	2
Quarterly	3 months	4
Monthly	1 month	12
Weekly	1 week	52

Table 1.2: Commonly Used Compounding Frequencies

**Example 1.1.1.** Find the periodic rate of interest for:

- a) 3% p.a. compounded quarterly
- b) 10% p.a. compounded weekly
- c) 8% p.a. compounded annually
- d) 35% p.a. compounded daily

**Solution:** In each problem we must write each percent as a decimal and identify the number of compounding periods per year.

- a) Here  $r = 0.03$  and  $m = 4$ , thus

$$\begin{aligned} i &= \frac{r}{m} \\ &= \frac{0.03}{4} \\ &= 0.0075 \\ &= 0.75\% \end{aligned}$$

b) Here  $r = 0.10$  and  $m = 52$ , thus

$$\begin{aligned}i &= \frac{r}{m} \\ &= \frac{0.10}{52} \\ &= 0.0019230769 \\ &= 0.19230769\%\end{aligned}$$

Here we should record many digits as our answer to avoid error in potential applications.

c) Here  $r = 0.08$  and  $m = 1$ , thus

$$\begin{aligned}i &= \frac{r}{m} \\ &= \frac{0.08}{1} \\ &= 0.08 \\ &= 8\%\end{aligned}$$

d) In our final part  $r = 0.25$  and  $m = 365$ , thus

$$\begin{aligned}i &= \frac{r}{m} \\ &= \frac{0.35}{365} \\ &= 0.095890411 \\ &= 9.5890411\%\end{aligned}$$

Again, we need to record many digits to reduce errors.

This concludes our problem. ◀

### 1.1.3 Compounding periods

We now draw our attention to the value  $n$  in Equation 1.3. We called  $n$  the number of compounding periods. In the financial industry it is often times the case that  $n$  is not outright given. The value  $n$  depends on  $m$  and the length of the term of the financial product or investment. This term length of time is usually given in terms of years. So if we have a financial product with term length  $t$  years and  $m$  compounding periods per year, then

$$n = mt \tag{1.6}$$

**Example 1.1.2.** Find the number of compounding periods for:

a) compounding semi-annually for 3 years

- b) compounding quarterly for 10 years
- c) compounding weekly for 40 months
- d) compounding daily for 320 months

**Solution:** In each problem we must identify the compounding frequency and multiply by the period in years.

- a) Here  $t = 3$  and  $m = 2$ , thus

$$\begin{aligned} n &= mt \\ &= (2)(3) \\ &= 6 \end{aligned}$$

- b) Here  $t = 10$  and  $m = 4$ , thus

$$\begin{aligned} n &= mt \\ &= (4)(10) \\ &= 40 \end{aligned}$$

- c) Here  $t = \frac{40}{12}$  and  $m = 52$ , thus

$$\begin{aligned} n &= mt \\ &= (52) \left( \frac{40}{12} \right) \\ &= 173.\bar{3} \end{aligned}$$

- d) In our final part  $t = \frac{320}{12}$  and  $m = 365$ , thus

$$\begin{aligned} n &= mt \\ &= (365) \left( \frac{320}{12} \right) \\ &= 9733.\bar{3} \end{aligned}$$

Note here that we do not round repeating decimals because this is not an application problem and we were not told to round the answers. In general, rounding should only occur at the very end of the problem, as rounding in the middle of your work will result in potentially significant errors. These errors will be penalized on tests and quizzes. ◀

### 1.1.4 Finding the compounding factor

Earlier, we discussed compounding factors. As a reminder the compounding factor in the compound interest formula is the expression  $(1 + i)^n$ , where  $i$  is the periodic rate of interest and  $n$  is the number of compounding periods. To avoid issues with the order of operations fundamental in arithmetic within this course it is highly recommended that students compute the compounding factor first. To accommodate this we work through a few examples.

**Example 1.1.3.** Compute the compounding factor  $(1 + i)^n$  for each of the following, round your answers to four decimal places.

- a) 3% p.a. compounded semi-annually for 2 years
- b) 7% p.a. compounded quarterly for 6 years
- c) 10% p.a. compounded weekly for 25 months
- d) 4.25% p.a. compounded daily for 2 decades

**Solution:** Here it will be important to understand the exact answer versus the computed answer. The exact expression/answer does not involve rounding or approximation. When you see the symbol,  $\approx$ , this means that some rounding has occurred and that the exact answer and computed answer are not exactly the same. In these problem, the exact answer is the last line with the symbol,  $=$ , next to it. In some situations, the exact answer and the computed answer will be the same. In application, we will always be looking for the computed answer.

We will also need to compute  $i$  using Equation 1.5 and  $n$  using Equation 1.6. We will integrate them directly into the compounding factor. Let us proceed.

- a) Here  $t = 2$ ,  $m = 2$ , and  $r = 0.03$  thus

$$\begin{aligned}(1 + i)^n &= \left(1 + \frac{r}{m}\right)^{mt} \\ &= \left(1 + \frac{0.03}{2}\right)^{2 \times 2} \\ &= (1 + 0.015)^4 \\ &= 1.015^4 \\ &= 1.06136355 \dots \\ &\approx 1.0614\end{aligned}$$

b) Here  $t = 6$ ,  $m = 4$ , and  $r = 0.07$  thus

$$\begin{aligned}(1+i)^n &= \left(1 + \frac{r}{m}\right)^{mt} \\ &= \left(1 + \frac{0.07}{4}\right)^{4 \times 6} \\ &= (1 + 0.0175)^{24} \\ &= 1.0175^{24} \\ &= 1.5164427 \dots \\ &\approx 1.5164\end{aligned}$$

c) Here  $t = \frac{25}{12}$ ,  $m = 52$ , and  $r = 0.10$  thus

$$\begin{aligned}(1+i)^n &= \left(1 + \frac{r}{m}\right)^{mt} \\ &= \left(1 + \frac{0.10}{52}\right)^{52 \times \frac{25}{12}} \\ &= (1 + 0.00192308 \dots)^{108.\bar{3}} \\ &= (1.00192308 \dots)^{108.\bar{3}} \\ &\approx 1.23137726 \dots \\ &\approx 1.2314\end{aligned}$$

d) In our final part  $t = 2 \times 10 = 20$ ,  $m = 365$ , and  $r = 0.0425$  thus

$$\begin{aligned}(1+i)^n &= \left(1 + \frac{r}{m}\right)^{mt} \\ &= \left(1 + \frac{0.0425}{365}\right)^{365 \times 20} \\ &= (1 + 0.000116438 \dots)^{7300} \\ &\approx (1.000116438 \dots)^{7300} \\ &\approx 2.33953108 \dots \\ &\approx 2.3395\end{aligned}$$

Some students may choose to compute  $i$  and  $n$  directly from Equations 1.5 and 1.6 and inserting these results into Equation 1.3 instead of adding the extra layer of mathematics into the work as has been done above. Students must make their own decision on how much detail to use. This concludes our example. ◀

### 1.1.5 Final conclusions and analytic construction

In summary, it should be noticed, from the example above, that we can substitute Equations 1.5 and 1.6 into Equation 1.3. This gives us the most general form of the

formula for computing compound interest.

$$S = P \left( 1 + \frac{r}{m} \right)^{mt} \quad (1.7)$$

It should be noted that this final form is not typically used in the financial community, but has great benefit in very general problems. Again, each student should make a choice as to which of these forms work better for them.

## 1.2 Application of the Compound Interest Formula

It should be noted that this corresponds to section 9.2 in the course textbook. Further, the section is divided into three parts with the textbook. Part 1 covers the same material discussed within these notes. Part 2 covers material related to how to use a programmable financial calculator to work these problems. This second part will not be discussed in these notes and will not be formally discussed in class because it is difficult to make sure each student has the same calculator. However, it is recommended that the student read about how to use their calculator to help move quickly through some of the problems. The instructor will always discuss each calculation in detail. Part 3 is on applications related to how to deal with situations where the interest rate, which we will discuss here.

### 1.2.1 Finding the future (maturity) value of an investment

The real interest of the previous section is finding the future (maturity) value of an investment. Thus we will dive right in with some examples.

**Example 1.2.1.** How much will an investment grow to if initially we invested \$3000 at 5% per annum compounded quarterly for seven years?

**Solution:** Here we begin by reminding ourselves that since we are looking for how much an investment will grow to, this means we are looking for a future value, and thus will need either Equation 1.4 or 1.7. Here we choose to use Equation 1.4 in conjunction with Equations 1.5 and 1.6, respectively. We choose this method because it is the standard method used by the financial community.

We need to find  $i$  first by noting that  $r = 0.05$  and  $m = 4$

$$\begin{aligned} i &= \frac{r}{m} && \text{Equation 1.5} \\ &= \frac{0.05}{4} \\ &= 0.0125 \end{aligned} \quad (1.8)$$

Next, to find  $n$ , noting that  $m = 4$  from our previous work and  $t = 7$ . Be reminded that

we should always make sure  $t$  is in units of years. In this case it is, so we can move on.

$$\begin{aligned}n &= mt && \text{Equation 1.6} \\ &= 4 \times 7 \\ &= 28\end{aligned} \tag{1.9}$$

We can now substitute the result (Equations 1.8 and 1.9) into Equation 1.4, with  $P = 3000$ , and remembering that  $S$  is the future value and the value we are trying to find.

$$\begin{aligned}S &= P(1 + i)^n && \text{Equation 1.4} \\ &= 3000(1 + 0.0125)^{28} \\ &= 4247.976910 \dots\end{aligned}$$

Rounding this to the nearest dollar and cent gives us a future value of \$4247.98. ◀

**Example 1.2.2.** What will be the worth of a deposit of \$8450 made into a retirement savings plan from March 1, 1995, to December 1, 2015, at 4.4% p.a. compounded semi-annually?

**Solution:** Again we are looking for the future value of an investment. We need to find  $i$  and  $n$  first.

First, to find  $i$  by noting that  $r = 0.044$  and  $m = 2$

$$\begin{aligned}i &= \frac{r}{m} && \text{Equation 1.5} \\ &= \frac{0.044}{2} \\ &= 0.022\end{aligned} \tag{1.10}$$

Next, to find  $n$ , noting that  $m = 2$ . Here it is not clear what  $t$  is. **Students should read the manual for their calculator, as most financial calculators have a function to find the number of days between two dates.** It has been determined that this time interval is 249 months, or  $t = \frac{249}{12} = 20.75$  years.

$$\begin{aligned}n &= mt && \text{Equation 1.6} \\ &= 2 \times 20.75 \\ &= 41.5\end{aligned} \tag{1.11}$$

We can now substitute the result (Equations 1.10 and 1.11) into Equation 1.4, with  $P = 8450$ .

$$\begin{aligned}S &= P(1 + i)^n && \text{Equation 1.4} \\ &= 8450(1 + 0.022)^{41.5} \\ &= 20848.21535 \dots\end{aligned}$$

Rounding this to the nearest dollar and cent gives us \$20,848.22. ◀

## 1.2.2 Applications involving changes to the interest rate or principal

**Example 1.2.3.** Edward deposited \$3402 into a financial instrument that earns interest at 7.25% p.a. compounded monthly for 3 years. At that time, the interest rate changes to 10% p.a. compounded quarterly. What is the value of the deposit 5 years after the change in the instrument's terms?

**Solution:** This problem must be worked in two parts because the terms change, where  $S_1$  is the future value for the first set of terms and  $S_2$  is the second set. To find  $S_1$  we let  $P = 3402$  with  $r = 0.0725$ ,  $m = 12$ , and  $t = 3$ . The second piece of this problem,  $S_2$ , we let  $P = S_1$ ,  $r = 0.10$ ,  $m = 4$ , and  $t = 5$  because we are looking for the value 5 years after the change in terms. So we have

$$\begin{aligned} S_1 &= P(1 + i)^n && \text{Equation 1.4} \\ &= 3402 \left( 1 + \frac{0.0725}{12} \right)^{12 \times 3} \\ &= 3402(1 + 0.0060416 \dots)^{36} \\ &= 3402(1.2421523 \dots) \\ &= 4225.802135 \dots \end{aligned}$$

So  $S_1 = 4225.802135 \dots$ . Notice that we do not round this result. **Rounding in the middle of an application problem will almost always result in errors. Students will be penalized on tests and quizzes for rounding errors.** Now to find  $S_2$ .

$$\begin{aligned} S_2 &= P(1 + i)^n \\ &= S_1(1 + i)^n \\ &= 4225.802135 \dots \left( 1 + \frac{0.10}{4} \right)^{4 \times 5} \\ &= 4225.802135 \dots (1 + 0.025)^{20} \\ &= 4225.802135 \dots (1.025)^{20} \\ &= 4225.802135 \dots (1.6386164 \dots) \\ &= 6924.468852 \dots \end{aligned}$$

Thus the value of the instrument 5 years after the change in terms will be \$6924.47. ◀

**Example 1.2.4.** Assume we have a debt of \$900 that builds interest at 7% compounded semi-annually from June 1, 2010, to December 1, 2012, and 10% compounded quarterly thereafter. Find the accumulated value of the debt on September 1, 2014.

**Solution:** In this problem we will need to be careful as time is given in units of days, however, notice that the days are the first days of months. So we can figure out  $t$  in terms of months. For example, notice that the time period for the first part of the term is 30 months. Thus in this problem we will need to convert from units of months to units of years by dividing by 12.

Let us begin by finding the value of the debit up through December 1, 2012. Using Equation 1.4 with  $P = 900$ ,  $r = 0.07$ ,  $m = 2$ , and  $t = \frac{30}{12} = 2.5$  we have the future value of the debit as

$$\begin{aligned} S &= P(1 + i)^n \\ &= 900 \left(1 + \frac{0.07}{2}\right)^{2 \times 2.5} \\ &= 1068.91768 \dots \end{aligned}$$

So the debit is valued at about \$1068.92 on December 1, 2012. Now for the rest of the period of the debt. As we can see, from the problem, the terms change after December 1, 2012 to  $r = 0.10$  and  $m = 4$ . We also need to find the amount of time (in years) that passes from December 1, 2012 to September 1, 2014. Again ignoring the days we have 21 months or  $t = \frac{21}{12} = 1.75$ . So letting  $P = 1068.91768 \dots$  from above, we have

$$\begin{aligned} S &= P(1 + i)^n \\ &= 1068.91768 \dots \left(1 + \frac{0.10}{4}\right)^{4 \times 1.75} \\ &= 1270.607218 \dots \end{aligned}$$

Giving us the value of the debt on September 1, 2014 as \$1270.61. ◀

The next problem is the same idea but we will need to treat the problem in many pieces and many changes will be made to the terms. Consider it a challenge problem that could show up on a test or quiz.

**Example 1.2.5.** A loan of \$10,000 is made to a business by a bank with custom terms. The loan is to be repaid by the business in pieces. First, the business will make a payment of \$3000 one year after the loan was taken out. Secondly, \$8000 will be repaid after 4 years. Next, after 6 years the business will repay \$700. A final payment will be made 7 years after the loan was taken out. Interest on the loan is 11% p.a. compounded monthly for the first 4 years of the loan and 13% compounded monthly after 4 years. How much should the final payment be?

**Solution:** We will have to work this problem gradually, some students will find a diagram helpful for organizing their work. Note that the first payment occurs one year into the loan term. So we will need to know how much is owed after this first payment using Equation 1.3. Thus if  $P = 10000$ ,  $r = 0.11$ ,  $m = 12$ , then the future value after  $t = 1$  year is

$$\begin{aligned} S_1 &= P(1 + i)^n \\ &= 10000 \left(1 + \frac{0.11}{12}\right)^{12 \times 1} \\ &= 11157.18836 \dots \end{aligned}$$

Now this is the future value of the loan after one year but note that the business pays off \$3000. Thus the value of the loan will now be  $11157.18836 - 3000 = \$8157.188362 \dots$

Now for the next payment. This payment takes place 4 years after the loan was taken out. We need to find the value of the loan after four years where  $P = 8157.188362 \dots$ . We will call it  $S_4$ . Also note that the time since this last payment is 3 years, otherwise the terms do not change. So  $t = 3$  in our work below.

$$\begin{aligned} S_4 &= P(1+i)^n \\ &= 8157.188362 \dots \left(1 + \frac{0.11}{12}\right)^{12 \times 3} \\ &= 11329.34459 \dots \end{aligned}$$

So the future value of the loan after 4 years and now removing the payment comes to  $11329.34459 - 8000 = \$3329.34459 \dots$ . This represents the amount owed to the bank after 4 years.

The next payment comes after 6 years (or 2 year later). Further notice that the interest rate changes to 13% after the fourth year. So we have  $P = 3329.34459 \dots$ ,  $r = 0.13$ ,  $m = 12$ , and  $t = 2$ . So

$$\begin{aligned} S_6 &= P(1+i)^n \\ &= 3329.34459 \dots \left(1 + \frac{0.13}{12}\right)^{12 \times 2} \\ &= 4311.8938 \dots \end{aligned}$$

So the value of the loan minus the payment on the 6th year is  $4311.8938 \dots - 700 = 3611.8938 \dots$

Finally, students should be cautious, as this is not the final answer because the final payment, whatever it is, will not be made until the 7<sup>th</sup> year of the loan (1 year after this \$700 payment). So we need to calculate the future value of this remaining amount, call it  $S_7$ .

$$\begin{aligned} S_7 &= P(1+i)^n \\ &= 3611.8938 \left(1 + \frac{0.13}{12}\right)^{12 \times 1} \\ &= 4110.4525 \dots \end{aligned}$$

Thus the value of the loan after 7 years under these payment terms is \$4110.45. This represents the final payment necessary to completely pay off the loan to the bank, concluding our problem. ◀

## 1.3 Present Value in Compound Interest

This material corresponds to section 9.3 in the textbook.

It is useful in many applications in finance to be able to find the present value or principal. Consider a situation where you may need to know how much money to initially invest over a given period of time, at a fixed rate, with a given compounding frequency to get a specified future value. Retirement works a little like this, although a bit more complex.

Within finance, the process of finding the present value or principal is called **discounting**. There are other terms, within finance, synonymous with principal and present value, such as **discounted value**, **proceeds**, and **compound discount**. You may encounter these terms later in your occupation. We will use the word, *principal*, from this point on unless other language is necessary.

We now derive a formula for quickly computing the principal for the purposes of applications. It is natural to start with a formula that we have already worked with such as Equation 1.4 and solve it for  $P$ . So

$$\begin{aligned} S &= P(1+i)^n && \text{Equation 1.4} \\ \frac{S}{(1+i)^n} &= \frac{P(1+i)^n}{(1+i)^n} && \text{Divide both sides by } (1+i)^n \\ \frac{S}{(1+i)^n} &= P && (1.12) \end{aligned}$$

or

$$S(1+i)^{-n} = P \quad \text{Equation 1.12 modified by exponent rules} \quad (1.13)$$

Either of Equations 1.12 or 1.13 will be fine to work applications, however, 1.12 is easier to use with simple calculators and 1.13 works better with scientific and more advanced calculators. If students are not sure which to use, it is recommended that your default formula be Equation 1.12. We will use Equation 1.12 in these notes from this point forward.

**Example 1.3.1.** Find the present value of \$1500 due in three years and eight months if money is worth 8% p.a. compounded quarterly.

**Solution:** Here we are told to look for  $P$ , when  $r = 0.08$ ,  $t = 44$  months  $= \frac{44}{12} = 3.\bar{6}$  years, and  $m = 4$ . We know  $S = 1500$  is the future value because the problem is referencing  $t$  as a time in the future. So

$$\begin{aligned} P &= \frac{S}{(1+i)^n} && \text{Equation 1.12} \\ &= \frac{1500}{(1+0.08/4)^{4 \times 3.\bar{6}}} \\ &= \frac{1500}{(1+0.02)^{14.\bar{6}}} \\ &= 1121.903251 \dots \end{aligned}$$

Thus the principal or present value of this financial instrument is \$1121.90. ◀

**Example 1.3.2.** What principal will need to be initially invested to amount to \$150,320 in 120 months at a nominal interest rate of 7.25% compounded daily?

**Solution:** Again we are looking for the principal. We know that the future value,  $S$ , is \$150,320 and  $t = 120$  months =  $\frac{120}{12} = 10$  years. The nominal rate of interest is the per annum rate, so  $r = 0.0725$  and  $m = 365$ .

$$\begin{aligned} P &= \frac{S}{(1+i)^n} && \text{Equation 1.12} \\ &= \frac{150320}{(1+0.0725/365)^{365 \times 10}} \\ &= \frac{150320}{(1+0.00019863\dots)^{3650}} \\ &= 72808.91001\dots \end{aligned}$$

So our principal should be \$72,808.91. ◀

**Example 1.3.3.** Find the present value of \$2000 due seven years from today if interest is 4.9% p.a. compounded monthly.

**Solution:** Again we are looking for the principal. We know that the future value,  $S = 2000$ ,  $t = 7$ , while  $r = 0.049$  and  $m = 12$ . So

$$\begin{aligned} P &= \frac{S}{(1+i)^n} && \text{Equation 1.12} \\ &= \frac{2000}{(1+0.049/12)^{12 \times 7}} \\ &= \frac{2000}{(1+0.00408\bar{3})^{84}} \\ &= 1420.268019\dots \end{aligned}$$

So the principal needed to generate \$2000 in seven years under these terms comes to \$1420.27. ◀

## 1.4 Finding $n$ and Related Problems

This section corresponds to section 10.1 in the textbook and is most likely given its own chapter in the book because the mathematics involved become more complicated. A significant review of roots, radicals, and logarithms is highly recommended at this time for students that are interested in an in depth understanding of the formulae used in the remaining sections of this chapter (chapter 10 in the textbook). The instructor will use as much time as is possible for reviewing the algebra, however, the course timeline does prevent much detail from being covered directly in class. It is also recommended that the student feel free to contact the instructor if confusion arises. The materials covered

in class will be more than enough detail to allow students to function well in the course from a finance perspective.

The questions arising in this section relate to  $n$ , the number of conversion (compounding) periods in a compound interest related financial instrument. These questions could involve finding  $n$  itself or more importantly, since  $n = mt$  (Equation 1.6), finding  $t$ . Thus it should come as no surprise that we may need to make use of Equation 1.6 at various points.

Before we begin formulae derivations we will use the following notation to shorten or work. We agree to use the notation RHS to mean the right hand side of an equation and LHS to mean the left hand side of an equation.

The first question we must answer is: How do we solve Equation 1.4 for  $n$ ? To answer this question, you will see, we must draw deeply into our algebra and advanced functions courses from secondary school. So

$$\begin{aligned} S &= P(1+i)^n && \text{Equation 1.4} \\ \frac{S}{P} &= \frac{P(1+i)^n}{P} && \text{Divide both sides by } P \\ \frac{S}{P} &= (1+i)^n && \text{Cancel out } P \text{ on RHS to become 1} \end{aligned} \quad (1.14)$$

The problem we have now is how to get  $n$  by itself. Notice  $n$  is in the exponent on the right hand side. As it turns out there is no simple algebraic tool to isolate  $n$  because it is apart of the exponent. So perhaps a way to remedy this would be to find a way to move  $n$  outside of the exponent. Logarithms is the solution here. Notice the following property of the *Natural Logarithm* from a functions course you may have taken:

$$\ln x^a = a \ln x \quad (1.15)$$

where  $x$  is any positive real number and  $a$  is any real number.

The critical issue students should get from this property is that when we have a logarithm of some expression with an exponent like the left hand side (LHS) of Equation 1.15 above, we can move the exponent out in front of the logarithm as a multiple. Effectively removing the exponent.

Now we know from algebra that we can do anything we want to an equation, for the most part, as long as we do it to both sides. Thus we propose taking natural logarithms of both sides of Equation 1.14. This gives us

$$\begin{aligned} \ln\left(\frac{S}{P}\right) &= \ln(1+i)^n \\ \ln\left(\frac{S}{P}\right) &= n \ln(1+i) && \text{Use Equation 1.15 on RHS} \\ \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} &= \frac{n \ln(1+i)}{\ln(1+i)} && \text{Divide both sides by } \ln(1+i) \\ \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} &= n && \ln(1+i)\text{'s cancel to give 1 on RHS} \end{aligned} \quad (1.16)$$

So now we have a formula for finding  $n$ . Note that when using this formula students will be pressing the natural logarithm button, (ln), on your calculator twice, not once. This is a common mistake based on faulty assumptions. Press it once after computing the fraction in the numerator and once after computation of the sum in the denominator.

We also mentioned above that many of our applications may need us to find the time,  $t$ . So how can we find  $t$  from Equations 1.16? The answer is to recall Equation 1.6 and note that we can solve it for  $t$ , once we have figured out  $n$  from Equation 1.16.

$$\begin{aligned} n &= mt && \text{Equation 1.6} \\ \frac{n}{m} &= t && \text{Divide both sides by } m \end{aligned} \quad (1.17)$$

So in other words, in an application problem related to compound interest where we are looking for  $t$ , we should first use Equation 1.16 to find  $n$ , then divide  $n$  by the compounding frequency  $m$  and that number will be  $t$  in units of years.

**Example 1.4.1.** How long will it take for \$5000 to grow to \$8250 at 5% compounded quarterly?

**Solution:** The problem is clearly asking for  $t$ , however, to find  $t$  we must first find  $n$  from Equation 1.16. So by letting  $P = 5000$ ,  $S = 8250$ ,  $i = \frac{0.05}{4} = 0.0125$  we have

$$\begin{aligned} n &= \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} \\ &= \frac{\ln\left(\frac{8250}{5000}\right)}{\ln(1+0.0125)} \\ &= \frac{\ln 1.65}{\ln 1.0125} \\ &= \frac{0.500775\dots}{0.012423\dots} \\ &= 40.31189\dots \text{ quarters} \end{aligned}$$

We now need to convert the quarters units to years using Equation 1.17 above.

$$\begin{aligned} t &= \frac{n}{m} \\ &= \frac{40.31189\dots}{4} \\ &= 10.07797\dots \text{ years} \end{aligned}$$

Thus it will take a little more than 10 years for \$5000 to grow to \$8250 under these terms. ◀

**Example 1.4.2.** How many years will it take for \$10,000, invested at 6% compounded monthly, to grow to \$20,000?

**Solution:** The problem is asking for  $t$  in years, however, to find  $t$  we must first find  $n$  just like the previous problem. So by letting  $P = 10000$ ,  $S = 20000$ ,  $i = \frac{0.06}{12} = 0.005$  we

have

$$\begin{aligned}n &= \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} \\&= \frac{\ln\left(\frac{20000}{10000}\right)}{\ln(1+0.005)} \\&= \frac{\ln 2}{\ln 1.005} \\&= \frac{0.69315\dots}{1.004988\dots} \\&= 138.97572\dots \text{ months}\end{aligned}$$

We now need to convert the month units to years using Equation 1.17 above.

$$\begin{aligned}t &= \frac{n}{m} \\&= \frac{138.97572\dots}{12} \\&= 11.58131\dots \text{ years}\end{aligned}$$

Thus it will take about 11.6 years for \$10,000 to grow to \$20,000 under these terms. ◀

Notice in this problem we were really looking for the length of time it take for the initial investment to grow to double its initial size. It is very important for students of finance to understand that the doubling, tripling, or quadrupling time of any investment does not depend on the initial size of the investment. In other words, if  $P_0$  is the initial investment, how long will it take for  $P_0$  to grow to  $2P_0$  or  $3P_0$  does not depend on  $P_0$ . Let us see why another look at example 1.4.2.

**Example 1.4.3.** How many years will it take for an initial investment, invested at 6% compounded monthly, to grow to double its initial amount?

**Solution:** First, note that this is effectively the same as the previous example problem except we are not given the present value or the future value.

The problem is asking for  $t$  in years, however, to find  $t$  we must first find  $n$  just like the previous problems. Since we do not know  $P$  we will let  $P = P_0$  where  $P_0$  is some initial investment. Now since we are looking for the time it takes for this initial amount to double, then the future value,  $S = 2P_0$ . Also,  $i$  remains the same from Example 1.4.2, so  $i = \frac{0.06}{12} = 0.005$ . Thus

$$\begin{aligned}n &= \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} \\&= \frac{\ln\left(\frac{2P_0}{P_0}\right)}{\ln(1+0.005)}\end{aligned}$$

Notice here that the  $P_0$ 's cancel out leaving only  $2(1) = 2$  within the logarithm of the numerator. This is a very important step, because looking below you will notice it leaves

us with exactly the same thing we had in Example 1.4.2 and the same answer.

$$\begin{aligned}
 &= \frac{\ln 2}{\ln 1.005} && P_0\text{'s cancel out} \\
 &= \frac{0.69315\dots}{1.004988\dots} \\
 &= 138.97572\dots \text{ months}
 \end{aligned}$$

We now need to convert the month units to years using Equation 1.17 above.

$$\begin{aligned}
 t &= \frac{n}{m} \\
 &= \frac{138.97572\dots}{12} \\
 &= 11.58131\dots \text{ years}
 \end{aligned}$$

Thus it will take about 11.6 years for any initial investment to grow to double its size under these terms. The importance of understanding this problem cannot be overestimated. ◀

**Example 1.4.4.** How long will it take in years for an investment to quadruple at 12% compounded quarterly?

**Solution:** Again, we are not given a future value or present value, so just make up a present value and multiply it by 4 to get the future value, because quadruple means to multiply by 4.

We are looking for  $t$  again, so to find  $t$  we must first find  $n$ . Let the present value  $P = P_0$  where  $P_0$  is some initial investment and the future value,  $S = 4P_0$ . Further,  $i = \frac{0.12}{4} = 0.03$ . Thus

$$\begin{aligned}
 n &= \frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} \\
 &= \frac{\ln\left(\frac{4P_0}{P_0}\right)}{\ln(1+0.03)} \\
 &= \frac{\ln 4}{\ln 1.03} && P_0\text{'s cancel out} \\
 &= \frac{1.38629\dots}{0.029559\dots} \\
 &= 46.89954\dots \text{ quarters}
 \end{aligned}$$

We now need to convert the quarter units to years using Equation 1.17 above.

$$\begin{aligned}
 t &= \frac{n}{m} \\
 &= \frac{46.89954\dots}{4} \\
 &= 11.724886\dots \text{ years}
 \end{aligned}$$

Thus it will take about 11.7 years for any initial investment to grow to quadruple its size under these terms. ◀

## 1.5 Finding $i$ and Related Problems

This section corresponds to section 10.2 in the course textbook. Here we will be interested in finding the periodic interest rate and nominal interest rate of a financial instrument involving compound interest.

Let us start by finding  $i$ , the periodic rate of interest and later we will think about the more important nominal rate. As you will see, we are going to need root and radical notation from algebra to work through much of this section. Recall

$$\begin{array}{ll}
 S = P(1 + i)^n & \text{Equation 1.4} \\
 \frac{S}{P} = \frac{P(1 + i)^n}{P} & \text{Divide both sides by } P \\
 \frac{S}{P} = (1 + i)^n & \text{On RHS the } P\text{'s cancel out} \\
 \sqrt[n]{\frac{S}{P}} = \sqrt[n]{(1 + i)^n} & \text{Take } n^{\text{th}} \text{ root of both sides} \\
 \sqrt[n]{\frac{S}{P}} = 1 + i & \text{On RHS the } n^{\text{th}} \text{ root and } n^{\text{th}} \text{ power cancel} \\
 \sqrt[n]{\frac{S}{P}} - 1 = i & \text{Subtract 1 from both sides} \quad (1.18)
 \end{array}$$

Equation 1.18 is one possible formula we can use to find the periodic interest rate. There is also an algebraically equivalent form

$$\left(\frac{S}{P}\right)^{\frac{1}{n}} - 1 = i \quad (1.19)$$

Either of these forms can be used effectively. Students will have to choose for themselves as to which they prefer. This will likely depend on the calculator used.

Now that we have formulae for the periodic rate of interest, how do we find the nominal or per annum rate of interest? The answer goes back to Equation 1.5. Where we want to solve for  $r$ , the nominal interest rate.

$$\begin{array}{ll}
 i = \frac{r}{m} & \text{Equation 1.5} \\
 im = r & \text{Multiply both sides by } m \quad (1.20)
 \end{array}$$

So this says, to find the nominal interest rate we should just multiply the result we get after using either Equation 1.18 or 1.19 with the compounding frequency.

**Example 1.5.1.** What is the annually compounded rate of interest if \$700 accumulates to \$1000 over 10 years?

**Solution:** Here the problem is clearly looking for  $r$  again, so to find  $r$  we must first find  $i$ . Let the present value be  $P = 700$ , the future value be  $S = 1000$ . Further, if

$m = 1$  because within the problem that interest will be annually compounded we have  $n = mt = (1)(10) = 10$ . So

$$\begin{aligned} i &= \sqrt[n]{\frac{S}{P}} - 1 && \text{Equation 1.18} \\ &= \sqrt[10]{\frac{1000}{700}} - 1 \\ &= 0.0363112\dots \end{aligned}$$

So the periodic rate of interest is about 3.6%.

Now to find the nominal or per annum interest rate. When calculating  $r$  make sure to use the exact figure above and not rounded figures. So we will use 0.0363112, not 0.036. Not doing so will result in penalizable errors on tests and quizzes.

$$\begin{aligned} r &= im && \text{Equation 1.20} \\ &= (0.0363112\dots)(1) \\ &= 0.0363112 \end{aligned}$$

So the nominal or per annum rate is the same as the periodic rate of about 3.6%. This should come as no surprise since the compounding frequency is once per year. ◀

**Example 1.5.2.** Find the nominal annual rate of interest compounded monthly if \$1200 accumulates to \$1618.62 in five years.

**Solution:** Here the problem is again looking for  $r$  again, so to find  $r$  we must first find  $i$ . Let the present value be  $P = 1200$ , the future value be  $S = 1618.62$ . Further, if  $m = 12$  because within the problem the interest will be compounded monthly, we have  $n = mt = (12)(5) = 60$ . So

$$\begin{aligned} i &= \sqrt[n]{\frac{S}{P}} - 1 && \text{Equation 1.18} \\ &= \sqrt[60]{\frac{1618.62}{1200}} - 1 \\ &= 0.004\bar{9} \end{aligned}$$

So the periodic rate of interest is about 0.5%.

Now to find the nominal or per annum interest rate.

$$\begin{aligned} r &= im && \text{Equation 1.20} \\ &= (0.004\bar{9})(12) \\ &= 0.058\bar{8} \end{aligned}$$

So the nominal or per annum rate is 5.88% or 5.9%. ◀

**Example 1.5.3.** At what nominal rate of interest compounded quarterly will money double in ten years?

**Solution:** Here the problem is again looking for  $r$  again, so to find  $r$  we must first find  $i$ . Let the present value be  $P = P_0$  because it is not given, the future value be  $S = 2P_0$ . Further, if  $m = 4$  because within the problem that interest will be compounded quarterly, we have  $n = mt = (4)(10) = 40$ . So

$$\begin{aligned}
 i &= \sqrt[n]{\frac{S}{P}} - 1 && \text{Equation 1.18} \\
 &= \sqrt[40]{\frac{2P_0}{P_0}} - 1 \\
 &= \sqrt[40]{2} - 1 && \text{The } P_0\text{'s cancel} \\
 &= 0.0174797\dots
 \end{aligned}$$

So the periodic rate of interest is about 1.7%.

Now to find the nominal or per annum interest rate.

$$\begin{aligned}
 r &= im && \text{Equation 1.20} \\
 &= (0.0174797\dots)(10) \\
 &= 0.17479692\dots
 \end{aligned}$$

So the nominal or per annum rate is about 17.5%. ◀

Students should be very careful when computing values in these past two sections as this material is extremely error prone. Please work carefully and ask questions if you are repeatedly making mistakes.

## 1.6 Effective and Equivalent Interest Rates

This section corresponds to section 10.3 in the text. We will begin motivating this section by doing an example related to the previous section.

**Example 1.6.1.** Suppose \$1000 earns interest of \$93.81 in one year.

- a) What is the nominal annual rate of interest compounded annually?
- b) What is the nominal annual rate of interest compounded monthly?

**Solution:** We will work part (a) first.

The problem is again looking for  $r$ , so to find  $r$  we must first find  $i$ . Let the present value be  $P = 1000$ , the future value be  $S = 1093.81$  (be careful here). Further, if  $m = 1$ ,

again, we are looking at only part (a) for now, we have  $n = mt = (1)(1) = 1$ . So

$$\begin{aligned}
 i &= \sqrt[n]{\frac{S}{P}} - 1 && \text{Equation 1.18} \\
 &= \sqrt[1]{\frac{1093.81}{1000}} - 1 \\
 &= \frac{1093.81}{1000} - 1 \\
 &= 0.09381
 \end{aligned}$$

So the periodic rate of interest is 9.381% This is exactly the same as the nominal rate because  $m = 1$ .

Now to work part (b).

On this part we are looking for  $r$ , so to find  $r$  we must first find  $i$ . Let the present value be  $P = 1000$ , the future value be  $S = 1093.81$ . In this case  $m = 12$ , again, we are looking at only part (b) for now, we have  $n = mt = (12)(1) = 12$ . So

$$\begin{aligned}
 i &= \sqrt[n]{\frac{S}{P}} - 1 && \text{Equation 1.18} \\
 &= \sqrt[12]{\frac{1093.81}{1000}} - 1 \\
 &= 0.007500238\dots
 \end{aligned}$$

So the periodic rate of interest in this part is about 0.75%. Now to find the annual or nominal rate.

$$\begin{aligned}
 r &= im && \text{Equation 1.20} \\
 &= (0.007500238\dots)(12) \\
 &= 0.09000286\dots
 \end{aligned}$$

So the nominal or per annum rate is about 9%. ◀

The point of this example was to have students notice that we can get the same amount of interest (\$93.81) with different rates and/or compounding frequencies. Since this is the case, it would be helpful if we could *modularize* (or standardize or make comparable) interest rates and compounding frequencies. Thus we have, what is called the **Effective Interest Rate**. The Effective Interest Rate is the rate of interest compounded annually (1 time per year). Thus in the problem above, the interest rate in part (a) is an effective interest rate (9.381%). When comparing terms of a financial instrument, where we are not sure which one generates more interest, we can find the effective interest rate of the instruments and take the one that is highest. This will assure us that we have chosen the best interest building instrument.

The natural question that arises is how do we compute the effective interest rate of a financial instrument? To answer this we must consider that we are given a periodic interest rate  $i$ , number of compounding periods  $n$ , present value  $P$ , and these values

must generate the same future value as the circumstances under the effective interest rate. So what are these effective interest rate circumstances? Well, the present value and compounding periods will be the same as the non-effective interest rate circumstances except for the interest rate and compounding frequency. See third equation below. Let us write it in mathematics. Let  $f$  be the effective interest rate. So

$$\begin{aligned}
 S &= P(1+i)^n && \text{Equation 1.4} \\
 S &= P\left(1 + \frac{r}{m}\right)^{mt} && \text{Substitute Equations 1.5 and 1.6} \\
 P\left(1 + \frac{f}{1}\right)^t &= P\left(1 + \frac{r}{m}\right)^{mt} && \text{Substitute ideas from above} \\
 P(1+f)^t &= P\left(1 + \frac{r}{m}\right)^{mt} \\
 (1+f)^t &= \left(1 + \frac{r}{m}\right)^{mt} && \text{Divide both sides by } P \\
 (1+f)^t &= \left(\left(1 + \frac{r}{m}\right)^m\right)^t && \text{Property of exponents} \\
 \sqrt[t]{(1+f)^t} &= \sqrt[t]{\left(\left(1 + \frac{r}{m}\right)^m\right)^t} && \text{Take } t \text{ root of both sides} \\
 1+f &= \left(1 + \frac{r}{m}\right)^m && \text{The } t \text{ root and } t \text{ powers cancel} \\
 f &= \left(1 + \frac{r}{m}\right)^m - 1 && \text{Subtract 1 from both sides} \quad (1.21)
 \end{aligned}$$

Thus we have our formula for the effective interest rate. Again, in this formula  $r$  is the rate of interest we wish to convert to an effective interest rate and  $m$  is the number of compounding periods in the term of the instrument. Some students may prefer a more familiar form, such as

$$f = (1+i)^m - 1 \quad (1.22)$$

where  $i$  is the periodic rate of interest. Students must choose which form they prefer.

**Example 1.6.2.** Find the effective rate of interest corresponding to 9% p.a. compounded semi-annually.

**Solution:** Here we use formula 1.21 or 1.22. We choose 1.22, where  $i = \frac{0.09}{2} = 0.045$  and  $m = 2$ .

$$\begin{aligned}
 f &= (1+i)^m - 1 \\
 &= (1+0.045)^2 - 1 \\
 &= 0.092025
 \end{aligned}$$

So the effective rate of interest is 9.2025%. ◀

**Example 1.6.3.** Find the effective rate of interest corresponding to 10% p.a. compounded monthly.

**Solution:** We use Equation 1.22 again where  $i = \frac{0.10}{12} = 0.008\bar{3}$  and  $m = 12$ .

$$\begin{aligned} f &= (1 + i)^m - 1 \\ &= (1 + 0.008\bar{3})^{12} - 1 \\ &= 0.10471307\dots \end{aligned}$$

So the effective rate of interest is about 10.47%. ◀

In fact we can convert any interest rate  $r_1$  and compounding frequency  $m_1$  to any other interest rate  $r_2$  and compounding frequency  $m_2$  given enough information. We give this formula without derivation because it is very similar to the derivation above for Equation 1.21 and is very much related.

$$\left(1 + \frac{r_1}{m_1}\right)^{m_1} = \left(1 + \frac{r_2}{m_2}\right)^{m_2} \quad (1.23)$$

With this formula substitute the values you have and solve for the variable that you don't have, using ideas we have used in class thus far. Consider the following example.

**Example 1.6.4.** Find the nominal annual rate compounded semi-annually that is equivalent to an annual rate of 6% compounded annually.

**Solution:** We use Equation 1.23 where  $r_1 = 0.06$ ,  $m_1 = 1$ , and  $m_2 = 2$ . We need to find  $r_2$ . So

$$\begin{aligned} \left(1 + \frac{r_1}{m_1}\right)^{m_1} &= \left(1 + \frac{r_2}{m_2}\right)^{m_2} \\ \left(1 + \frac{0.06}{1}\right)^1 &= \left(1 + \frac{r_2}{2}\right)^2 \\ 1 + 0.06 &= \left(1 + \frac{r_2}{2}\right)^2 \\ 1.06 &= \left(1 + \frac{r_2}{2}\right)^2 \end{aligned}$$

Remember, the goal is to isolate  $r_2$ . Proceeding from above we will take the square root of both sides to get rid of the square on the right hand side.

$$\begin{aligned} \sqrt{1.06} &= \sqrt{\left(1 + \frac{r_2}{2}\right)^2} \\ 1.029563014\dots &= 1 + \frac{r_2}{2} && \text{Root and square cancel on RHS} \\ 2(1.029563014\dots) &= 2\left(1 + \frac{r_2}{2}\right) && \text{Multiply both sides by 2} \\ 2.059126\dots &= 2 + 2\frac{r_2}{2} && \text{Distribute the 2 on RHS} \\ 2.059126\dots &= 2 + r_2 && \text{Cancel 2's on RHS} \\ 0.059126\dots &= r_2 && \text{Subtract 2 from both sides} \end{aligned}$$

Thus the equivalent nominal annual rate compounded semi-annually is about 5.9126%. ◀

**Example 1.6.5.** Roy's bank currently offers 2.4% compounded semi-annually on premium savings deposits. His friend has told him that he would earn more interest if his savings compounded more frequently. What is the nominal annual rate compounded monthly that would be equivalent to his semi-annual rate? [Suggested Strategy: Convert Nominal Rate compounded semi-annually to Effective Rate and then back to a Nominal rate compounded monthly]

**Solution:** We use Equation 1.23 again where  $r_1 = 0.024$ ,  $m_1 = 2$ , and  $m_2 = 12$ . We need to find  $r_2$ . So

$$\begin{aligned} \left(1 + \frac{r_1}{m_1}\right)^{m_1} &= \left(1 + \frac{r_2}{m_2}\right)^{m_2} \\ \left(1 + \frac{0.024}{2}\right)^2 &= \left(1 + \frac{r_2}{12}\right)^{12} \\ 1.024144 &= \left(1 + \frac{r_2}{12}\right)^{12} \end{aligned}$$

Again, the goal is to isolate  $r_2$ . Proceeding from above we will take the 12<sup>th</sup> root of both sides to get rid of the 12<sup>th</sup> power on the right hand side.

$$\begin{aligned} \sqrt[12]{1.024144} &= \sqrt[12]{\left(1 + \frac{r_2}{12}\right)^{12}} && \text{Root and exponent cancel on RHS} \\ 1.00199007\dots &= 1 + \frac{r_2}{12} && \\ 12(1.0199007\dots) &= 12\left(1 + \frac{r_2}{12}\right) && \text{Multiply both sides by 12} \\ 12.023881\dots &= 12 + 12\frac{r_2}{12} && \text{Distribute the 12 on RHS} \\ 12.02388\dots &= 12 + r_2 && \text{Cancel 12's on RHS} \\ 0.02388\dots &= r_2 && \text{Subtract 12 from both sides} \end{aligned}$$

Thus the equivalent nominal annual rate compounded monthly is about 2.388%.

There is an important point to this problem. The friend is correct in saying that given the same interest rates its better to have higher compounding frequency. ◀

## 2 Ordinary Simple Annuities

This chapter corresponds to Chapter 11 in the course textbook and covers the simplest class of annuities. We begin with a basic introduction of what an annuity is and how to classify them.

### 2.1 Introduction to Annuities and Classification

#### 2.1.1 Basic concepts

An **annuity** is a series of payments, usually of equal size, made at periodic intervals. The length of time between successive payments is called the **payment interval** or **payment period**. The length of time from the beginning of the first payment until the end of the final payment is called the **term of the annuity**. The size of the regular payment is sometimes called the **periodic rent**.

Students should work hard to memorize the following annuity classification scheme. Much of the problem solving in this course from this point on will depend on the student's ability to quickly classify financial instruments.

#### 2.1.2 Types of annuities

1. Simple and general annuities

Annuities are classified by the length of the conversion period relative to the payment period. With a **simple annuity**, the compounding (conversion) period is the same as the payment interval. An example might be monthly payments on a loan where the interest is compounded monthly. Since the monthly payments per year is 12 and the compounding periods per year is also 12, thus we have a simple annuity.

With a **general annuity**, the conversion period and the payment interval are not the same. An example of such a situation might be a residential mortgage, where interest is compounded semi-annually but payments are made monthly.

2. Ordinary annuities and annuities due

Annuities are classified further by the date of payment. In an **ordinary annuity**, payments are made at the end of each payment period. In an **annuity due**, payments are made at the beginning of each payment period. Loan payments, mortgage payments, and interest payments on bonds are all examples of ordinary annuities. Examples of annuities due include lease rental payments on real estate or equipment.

3. Deferred annuities

A **deferred annuity** occurs when the first payment is delayed for a period of time. For example, a severance amount may be deposited into a fund that earns interest, and then later converted into another fund that earns interest, and then later converted into another fund that pays out a series of payments until the fund is exhausted.

4. Perpetuities

A **perpetuity** is an annuity for which the payments continue forever. When the size of the periodic payment from a fund is equal to or less than the periodic interest earned by the fund, such as an endowment fund to a university or a continuous benefit from a capital investment, a perpetuity is the result.

5. Annuities certain and contingent annuities

If both the beginning date and the ending date of an annuity are known, indicating a fixed term, the classification is an **annuity certain**. Examples include lease payments, instalment payments on loans, and interest payments on bonds.

If the beginning date, the ending date, or both, are unknown, the classification is a **contingent annuity**. Life insurance premiums or pension payments are typical examples of contingent annuities.

Question	If answer is YES:	If answer is NO:
Is there a series of equal periodic payments?	This is an annuity	This is not an annuity
Is the compounding period equal to the payment interval?	Simple annuity	General annuity
Is the payment made at the end of the payment period?	Ordinary annuity	Annuity due
Does the payment start in the first payment period?	Not deferred	Deferred annuity
Are the beginning and ending dates known?	Annuity certain	Perpetuity, if term is continuous; contingent annuity, if dates are not know

Table 2.1: Classification of Annuities

**Example 2.1.1.** Natasha has an agreement with a friend to repay a loan. The agreement states that she will start payments of \$50 at the end of this month and continue each month for 3 years at an interest rate of 7% per annum compounded monthly. Clas-

sify this annuity by (a) term; (b) date of payment; and (c) conversion (compounding) period.

**Solution:** This is clearly an annuity because there is a constant and regular payment involved.

- a) It is annuity certain because the beginning and end of the term of the annuity is clearly stated.
- b) It is an ordinary annuity because she will make the payments at the end of the month.
- c) It is a simple annuity because both the compounding period and the payment period are both 12 times per year (monthly).

This concludes our problem. ◀

**Example 2.1.2.** Alice's grandfather passed and left a considerable estate. In his will Alice was directed to place \$5 million into a bond account paying 3.5% compounded quarterly. After the money has set in the account for 5 years, the interest generated each year will be given away as a scholarship. The \$5 million principal is to be left in the account indefinitely for this purpose. Classify this annuity by (a) term; (b) date of payment; and (c) conversion (compounding) period.

**Solution:** Again, this is clearly an annuity because of the scholarship payouts that will begin in the future.

- a) It is a perpetuity because the term is to continue indefinitely.
- b) It is an ordinary annuity because the scholarships will be paid out at the end of the year after the interest has accumulated. Note that this annuity is initially deferred.
- c) It is a general annuity because the compounding period is quarterly and the payment period each year.

This concludes our problem. ◀

**Example 2.1.3.** Deposits of \$25 are made at the beginning of each week into an account earning interest at 5% compounded monthly for 10 years. Classify this annuity by (a) term; (b) date of payment; and (c) conversion (compounding) period.

**Solution:** Again, we are clearly dealing with an annuity.

- a) It is annuity certain because the beginning and end of the term of the annuity is clearly stated.
- b) It is annuity due because the payment is made at the beginning of each week.
- c) It is a general annuity because the compounding period is monthly and the payment period each week.

This concludes our problem. ◀

## 2.2 Finding the Future Value of an Ordinary Simple Annuity

This section corresponds to section 11.2 in the course textbook and deals with problems where we are making regular payments into a financial account and are interested in finding the accounts value at some time in the future. To avoid potential confusion that may result from formulae on tests and within the book we will now begin to adopt the following notation,  $FV$ , for future value instead of  $S$  used in the previous chapter.

We begin to motivate the future value of an ordinary simple annuity with an example.

**Example 2.2.1.** Find the future value of 6 equal deposits of \$100 made at the end of each year for the next six years at 4% compounded annually, just after the final deposit has been made.

**Solution:** This is clearly an ordinary simple annuity because the payments are made each year and the compounding period is annually and the payments are made at the end of each year. The trick to this problem is finding the future value of each of the 6 deposits individually, then add them all up. To do this we will use Equation 1.4 from the previous chapters on each of the payments.

- Deposit 1

Made at the end of year 1. It will accumulate for 5 years, thus  $n = 5$ ,  $i = 0.04$ ,  $PV = 100$ , thus

$$\begin{aligned}FV &= PV(1 + i)^n \\&= 100(1 + 0.04)^5 \\&= 100(1.04)^5 \\&= \$121.67\end{aligned}$$

- Deposit 2

Made at the end of year 2. It will accumulate for 4 years, thus  $n = 4$ ,  $i = 0.04$ ,  $PV = 100$ , thus

$$\begin{aligned}FV &= PV(1 + i)^n \\&= 100(1 + 0.04)^4 \\&= 100(1.04)^4 \\&= \$116.99\end{aligned}$$

- Deposit 3

Made at the end of year 3. It will accumulate for 3 years, thus  $n = 3$ ,  $i = 0.04$ ,  $PV = 100$ , thus

$$\begin{aligned}FV &= PV(1 + i)^n \\&= 100(1 + 0.04)^3 \\&= 100(1.04)^3 \\&= \$112.49\end{aligned}$$

- Deposit 4  
Made at the end of year 4. It will accumulate for 2 years, thus  $n = 2$ ,  $i = 0.04$ ,  $PV = 100$ , thus

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= 100(1 + 0.04)^2 \\ &= 100(1.04)^2 \\ &= \$108.16 \end{aligned}$$

- Deposit 5  
Made at the end of year 5. It will accumulate for 1 years, thus  $n = 1$ ,  $i = 0.04$ ,  $PV = 100$ , thus

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= 100(1 + 0.04)^1 \\ &= 100(1.04)^1 \\ &= \$104 \end{aligned}$$

- Deposit 6  
Made at the end of year 6. It will accumulate for 0 years, thus  $n = 0$ ,  $i = 0.04$ ,  $PV = 100$ , thus

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= 100(1 + 0.04)^0 \\ &= 100(1.04)^0 \\ &= 100(1) && \text{Any number raised to the zero power is 1} \\ &= \$100 \end{aligned}$$

If we now add these up we get

$$121.67 + 116.99 + 112.49 + 108.16 + 104 + 100 = 663.31$$

So the future value of this annuity is \$663.31. ◀

### 2.2.1 Formula for the future value of an ordinary simple annuity

The previous problem in itself is nothing special. What you should notice is the second line of each computation in the work. They have the form  $PV(1 + i)^n$  where  $n$  starts at 0 and increases by 1 for each payment/conversion period up to 5 (notice not to 6). Thus for  $n$  conversion periods on an ordinary simple annuity we have a future value of

$$\begin{aligned} FV_n &= PV(1 + i)^0 + PV(1 + i)^1 + PV(1 + i)^2 + \cdots + PV(1 + i)^{n-1} \\ &= PV [(1 + i)^0 + (1 + i)^1 + (1 + i)^2 + \cdots + (1 + i)^{n-1}] \end{aligned} \quad (2.1)$$

This might not look useful in itself but the above equation is a special case in mathematics called a *geometric series* and can be simplified according to the following

$$r^0 + r^1 + r^2 + \dots + r^{n-1} = \frac{1 - r^n}{1 - r} \quad \text{For some positive number } n$$

Noting that if  $r = 1 + i$  in the formula above we get exactly the expression in the brackets of Equation 2.1. Thus starting from Equation 2.1 and continuing we have

$$\begin{aligned} FV_n &= PV \left[ \frac{1 - (1 + i)^n}{1 - (1 + i)} \right] \\ &= PV \left[ \frac{1 - (1 + i)^n}{-i} \right] && \text{Clean up the denominator} \\ &= PV \left[ \frac{1 - (1 + i)^n}{-i} \times \frac{-1}{-1} \right] \\ &= PV \left[ \frac{(1 + i)^n - 1}{i} \right] \end{aligned}$$

This is a generally good form for finding the future value of an ordinary simple annuity quickly. In fact, much quicker than the process used in Example 2.2.1. However we will change things a little to make the variables more appropriate for annuities. In particular, notice that  $PV$  in any annuity is a payment. So instead we replace  $PV$  with  $PMT$  and use  $FV_n$  for the future value of an ordinary simple annuity giving us

$$FV_n = PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \quad (2.2)$$

where  $n$  is the number of conversion (payment or compounding) periods and  $i$  is the periodic interest rate. The expression  $\frac{(1+i)^n - 1}{i}$  is called the **accumulation factor**.

**Example 2.2.2.** Find the accumulated value of monthly payments of \$75 for 30 years into an account that returns 8% compounded monthly.

**Solution:** Here we are dealing with finding the future value of an ordinary simple

annuity where  $PMT = 75$ ,  $i = \frac{0.08}{12} = 0.00\bar{6}$ , and  $n = 12(30) = 360$ . So

$$\begin{aligned}
 FV_n &= PMT \left[ \frac{(1+i)^n - 1}{i} \right] \\
 &= 75 \left[ \frac{(1 + 0.00\bar{6})^{360} - 1}{0.00\bar{6}} \right] \\
 &= 75 \left[ \frac{(1.00\bar{6})^{360} - 1}{0.00\bar{6}} \right] \\
 &= 75 \left[ \frac{(10.935727\dots) - 1}{0.00\bar{6}} \right] \\
 &= 75 \left[ \frac{9.935727\dots}{0.00\bar{6}} \right] \\
 &= 75(1490.359072\dots) \\
 &= 111776.9304\dots
 \end{aligned}$$

Thus the annuity accumulates a total of \$111,776.93 over 30 years. ◀

We will omit some of the details in the rest of the examples as they just involve issues with the order of operations. Ask questions if anything is unclear.

**Example 2.2.3.** Ammad deposits \$5 at the end of each week for 7 years into an account paying 4% compounded weekly. What will be the balance in the account at the end of the seven year term? How much of the balance was contributed by Ammad? How much is interest?

**Solution:** Here we are dealing with several questions related to an ordinary simple annuity where  $PMT = 5$ ,  $i = \frac{0.04}{52} = 0.000769231\dots$ , and  $n = 52(7) = 364$ .

First, to find the balance we use Equation 2.2. So

$$\begin{aligned}
 FV_n &= PMT \left[ \frac{(1+i)^n - 1}{i} \right] \\
 &= 5 \left[ \frac{(1 + 0.000769231\dots)^{364} - 1}{0.000769231\dots} \right] \\
 &= 5 \left[ \frac{(1.000769231\dots)^{364} - 1}{0.000769231\dots} \right] \\
 &= 5(414.6018523\dots) \\
 &= 2099.418114\dots
 \end{aligned}$$

Thus the annuity's balance is \$2099.42 over 7 years.

Secondly, to answer the question of how much did Ammad contribute, we must figure out how much he paid into the account over the seven years. We know that he made 364 payments of \$5. That is  $364 \times 5 = \$1820$  contributed by Ammad.

Finally, to answer the question of how much of the balance is interest alone, we must subtract the two values. So  $2099.42 - 1820 = \$279.42$  in interest. This concludes the example. ◀

**Example 2.2.4.** Sahar decides to begin saving money for a big purchase she intends to make in a few years. Thus she has set up an account with Scotiabank where she will deposit \$160 at the end of each month per month for 5 years at a rate of 3% compounded monthly. At the end of the five year period Sahar is surprised to find out that Bank of Montreal will offer her an account that will return 7% compounded quarterly on here savings. If Sahar decides to keep the money in the Scotiabank account for the 5 year duration of the annuity and then move to the 7% account at Bank of Montreal, how much money will Sahar have accumulated, if she just leaves the money in the BMO account for another 4 years?

**Solution:** There are two parts to this application problem. The first part is the annuity at Scotiabank, where  $PMT = 160$ ,  $i = \frac{0.03}{12} = 0.0025$ , and  $n = 12 \times 5 = 60$ . The second part, at BMO, is not an annuity because the problem says she will just leave the money in the account, meaning there will be no payments, thus not an annuity. In this BMO account we will simply need to find the future value of the balance of the 5 year annuity over another 4 years at 7% compounded quarterly. Here we will use Equation 1.4 again, where  $PV = \text{Balance from the annuity}$ ,  $i = \frac{0.07}{4} = 0.0175$ , and  $n = 4 \times 4 = 16$ .

Let us calculate the balance of the Scotiabank annuity first. So

$$\begin{aligned} FV_n &= PMT \left[ \frac{(1+i)^n - 1}{i} \right] \\ &= 160 \left[ \frac{(1+0.0025)^{60} - 1}{0.0025} \right] \\ &= 10343.47402\dots \end{aligned}$$

So the balance of the annuity at the end of the 5 year Scotiabank term is \$10343.47.

Now to finding the future value of this money, which will be transferred to the BMO account at 7% compounded quarterly. Remember, the money is no longer in an annuity. So

$$\begin{aligned} FV &= PV(1+i)^n && \text{Equation 1.4} \\ &= 10343.47402\dots (1+0.0175)^{16} \\ &= 13652.65495\dots \end{aligned}$$

So the balance in the account after another 4 years in the BMO account is \$13,652.65. ◀

## 2.3 Finding the Present Value of an Ordinary Simple Annuity

This section corresponds to section 11.3 in the textbook and provides the student of finance with tools that can be used to evaluate amortization of loans and similar topics that we will encounter later in the course. The general question is: how do we find the present value of a set of payments in an ordinary simple annuity? To illustrate the idea we work an example.

**Example 2.3.1.** Find the amount of money whose value now is the same as regular payments of \$1200 at the end of each year at 6% compounded annually for the next 5 years.

**Solution:** This is basically five separate problems using Equation 1.12 or 1.13 (we will use 1.13), the present value of money formula in the previous chapter. Here  $FV = 1200$ ,  $i = 0.06$ , and  $n$  will vary from 1 to 5. Lets have a look at each individual payment.

- Payment 1  
Here  $n = 1$ . So

$$\begin{aligned} PV &= FV(1+i)^{-n} && \text{Equation 1.13} \\ &= 1200(1+0.06)^{-1} \\ &= 1132.075472\dots \end{aligned}$$

Thus the first payment has a present value of about \$1132.08.

- Payment 2  
Here  $n = 2$ . So

$$\begin{aligned} PV &= FV(1+i)^{-n} \\ &= 1200(1+0.06)^{-2} \\ &= 1067.995728\dots \end{aligned}$$

Thus the second payment has a present value of about \$1068.00.

- Payment 3  
Here  $n = 3$ . So

$$\begin{aligned} PV &= FV(1+i)^{-n} \\ &= 1200(1+0.06)^{-3} \\ &= 1007.54314\dots \end{aligned}$$

Thus the third payment has a present value of about \$1007.54.

- Payment 4  
Here  $n = 4$ . So

$$\begin{aligned} PV &= FV(1+i)^{-n} \\ &= 1200(1+0.06)^{-4} \\ &= 950.5123959\dots \end{aligned}$$

Thus the fourth payment has a present value of about \$950.51.

- Payment 5  
Here  $n = 5$ . So

$$\begin{aligned} PV &= FV(1+i)^{-n} \\ &= 1200(1+0.06)^{-5} \\ &= 896.7098074\dots \end{aligned}$$

Thus the fifth payment has a present value of about \$896.71.

So, if we now add up all of the exact figures above we have  $1132.075472 + 1067.995728 + 1007.54314 + 950.5123959 + 896.7098074 = 5054.836543$  or \$5054.84 as the total present value of the 5 payments. ◀

### 2.3.1 Formula for the present value of an ordinary simple annuity

The following discussion is optional and meant to help students understand how the preceding formula came to be generated. Some students may find the reasoning somewhat challenging. A significant mathematics background is necessary.

Let  $PV_n$  be the present value of an ordinary simple annuity with  $n$  periodic payments, periodic interest rate of  $i$  where the periods of payment and the conversion (compounding) periods are the same, and payments of  $PMT$  made at the end of each period; then by looking at the second line in each of the payment equations in the above problem we can see that each payment's present value has the form  $FV(1+i)^{-n}$  but remember  $FV$  is really a payment so we will replace this with  $PMT$  and  $n$  will vary from 1 to the total number of periodic payments  $n$ , then we will add them all up. This gives us

$$\begin{aligned} PV_n &= PMT(1+i)^{-1} + PMT(1+i)^{-2} + PMT(1+i)^{-3} + \dots + PMT(1+i)^{-n} \\ &= \frac{PMT}{(1+i)^1} + \frac{PMT}{(1+i)^2} + \dots + \frac{PMT}{(1+i)^n} \end{aligned} \quad (2.3)$$

As it turns out we can multiply both sides of an equation by a number. Here we choose to multiply both sides of Equation 2.3 with  $\frac{1}{1+i}$ . Some students will wonder: Why do this?. Answer: watch carefully what happens after we do more mathematics. This gives us

$$\begin{aligned} \frac{1}{1+i}PV_n &= \frac{1}{1+i} \left[ \frac{PMT}{(1+i)^1} + \frac{PMT}{(1+i)^2} + \dots + \frac{PMT}{(1+i)^n} \right] \\ \frac{1}{1+i}PV_n &= \frac{PMT}{(1+i)^2} + \frac{PMT}{(1+i)^3} + \dots + \frac{PMT}{(1+i)^{n+1}} \end{aligned} \quad (2.4)$$

Next, we will subtract Equation 2.4 from Equation 2.3. Giving us

$$\begin{aligned} PV_n - \frac{1}{1+i}PV_n &= \frac{PMT}{(1+i)^1} + \frac{PMT}{(1+i)^2} + \frac{PMT}{(1+i)^3} + \dots + \frac{PMT}{(1+i)^n} \\ &\quad - \frac{PMT}{(1+i)^2} - \frac{PMT}{(1+i)^3} + \dots - \frac{PMT}{(1+i)^n} - \frac{PMT}{(1+i)^{n+1}} \end{aligned}$$

Notice that the aligned columns of mathematics become zero when subtracted. This leaves us with only the first and last terms. So

$$\begin{aligned}
 PV_n - \frac{1}{1+i}PV_n &= \frac{PMT}{1+i} - \frac{PMT}{(1+i)^{n+1}} \\
 PV_n \left(1 - \frac{1}{1+i}\right) &= PMT \left[ \frac{1}{1+i} - \frac{1}{(1+i)^{n+1}} \right] && \text{Factor out } PV_n \text{ on LHS and } PMT \text{ on RHS} \\
 PV_n \left(\frac{i}{1+i}\right) &= PMT \left[ \frac{1}{1+i} - \frac{1}{(1+i)^{n+1}} \right] && \text{Common denominator on LHS} \\
 \frac{PV_n \left(\frac{i}{1+i}\right)}{\left(\frac{i}{1+i}\right)} &= PMT \frac{\left[ \frac{1}{1+i} - \frac{1}{(1+i)^{n+1}} \right]}{\left(\frac{i}{1+i}\right)} && \text{Divide both sides by } \frac{i}{1+i} \\
 \frac{PV_n \left(\frac{i}{1+i}\right)}{\left(\frac{i}{1+i}\right)} &= PMT \frac{\left[ \frac{1}{1+i} - \frac{1}{(1+i)^{n+1}} \right]}{i \left(\frac{1}{1+i}\right)} && \text{Rewrite the denominator on RHS} \\
 \frac{PV_n \left(\frac{i}{1+i}\right)}{\left(\frac{i}{1+i}\right)} &= PMT \frac{\frac{1}{1+i} - \frac{1}{(1+i)^{n+1}}}{i \left(\frac{1}{1+i}\right)} && \text{Rewrite as two fractions on RHS} \\
 PV_n &= PMT \left[ \frac{1}{i} - \frac{1}{(1+i)^n} \right] && \text{Cancel out common factors} \\
 PV_n &= PMT \left[ \frac{1}{i} - \frac{1}{i(1+i)^n} \right] && \text{Reduce the complex fraction} \\
 PV_n &= PMT \left[ \frac{1}{i} - \frac{(1+i)^{-n}}{i} \right] && \text{Rearrange using laws of exponents} \\
 PV_n &= PMT \left[ \frac{1 - (1+i)^{-n}}{i} \right] && \text{Add fractions} \tag{2.5}
 \end{aligned}$$

This is the formula for the present value of an ordinary simple annuity. The expression  $\frac{1-(1+i)^{-n}}{i}$  is called the **discounting factor**.

**Example 2.3.2.** Find the present value at the beginning of the first payment period of payments of \$75 made at the end of every 6 months for 20 years, if interest is 7% compounded semi-annually?

**Solution:** Here the terms are rather obvious. But note that the payments are made at the end of a six month period, which gives us two payments per year and matches the conversion (compounding) period,  $m = 2$ . So with  $PMT = 75$ ,  $i = \frac{0.07}{2} = 0.035$ ,

and  $n = 2 \times 20 = 40$  we have

$$\begin{aligned}PV_n &= PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \\&= 75 \left[ \frac{1 - (1 + 0.035)^{-40}}{0.035} \right] \\&= 10343.47402 \dots\end{aligned}$$

Thus the present value is \$1601.63. ◀

**Example 2.3.3.** Suppose you want to withdraw \$100 at the end of each month for five years from an account paying 4.5% compounded monthly

- How much must you have on deposit at the beginning of the month in which the first withdrawal is made at the end of the month?
- How much will you receive in total?
- How much of what you will receive is interest?

**Solution:** Let us work part (a) first. Here we have  $PMT = 100$ ,  $i = \frac{0.045}{12} = 0.00375$ , and  $n = 12 \times 5 = 60$  we have

$$\begin{aligned}PV_n &= PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \\&= 100 \left[ \frac{1 - (1 + 0.00375)^{-60}}{0.00375} \right] \\&= 5363.93803 \dots\end{aligned}$$

Thus the initial amount in the account must be about \$5363.94 to sustain these payments for the term.

Now to part (b). To find out how much you will receive in total just multiply \$100 per month for 60 months,  $100 \times 60 = \$6000$ .

Finally part (c). We know that the initial balance must be \$5363.94, thus the interest received will be the difference between this number and the answer we received in part (b), so  $6000 - 5363.94 = \$636.06$ . ◀

### 2.3.2 Applications

There are many applications of the present value formula for simple ordinary annuities, however one of the more typical applications involves purchasing a home or property. We will cover mortgages and amortization in more detail in later chapters. Here will give this topic some elementary analysis.

When purchasing a home with a mortgage there is typically an initial **down payment**. The larger the down payment, the smaller the amount that needs to be borrowed. The amount to be borrowed (the loan) is the present value of the future periodic payments.

Thus we can use Equation 2.5 to find the loan amount if we know the terms of the loan and the payment amounts. We can also easily deduce that the value of an amortized property is given by

$$\text{Cash Value} = \text{Down Payment} + PV_n \quad (2.6)$$

**Example 2.3.4.** Steve and Selma have recently purchased a home with a \$30,000 down payment and are making payments of \$2000 at the end of each month for the next 15 years. Assume that their mortgage terms include a fixed rate of 4% p.a. compounded quarterly. How much was the property sold for? How much will they pay in total for the property? How much of what they paid is interest alone?

**Solution:** To figure out how much the property sold for we will need to use Equations 2.5 and 2.6. We use Equation 2.5 first with  $PMT = 2000$ ,  $i = \frac{0.04}{4} = 0.01$ , and  $n = 4 \times 15 = 60$  we have

$$\begin{aligned} PV_n &= PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \\ &= 2000 \left[ \frac{1 - (1 + 0.01)^{-60}}{0.01} \right] \\ &= 89910.07681 \dots \end{aligned}$$

Thus the loan was \$89,910.08. So using Equation 2.6 we can figure out the value of the property by adding this amount to 30000, which gives us  $89910.08 + 30000 = \$119,910.08$

The second question asks us how much total will Steve and Selma pay for the home. Well they initially paid \$30,000 and then paid \$2000 each month for 15 years that comes to  $30000 + 2000(12)(15) = \$390,000$ .

The final question asks how much of this amount was pure interest. Here we just need to take the difference of the first two questions. So  $390000 - 119910.08 = \$270,089.92$ . This concludes our problem. ◀

## 2.4 Finding the Periodic Payment of an Ordinary Simple Annuity

This section corresponds to section 11.4 in the course textbook and discusses how to find the periodic payment,  $PMT$  in an ordinary simple annuity. In this section the problems will come in basically two forms. Form 1 where you are given the future value of the annuity with the periodic interest rate and number periods. And Form 2 where you are given the present value of the annuity with the periodic interest rate and number of periods. We will cover each of these topics with subsections a little later on. First we need to figure out how to find the payment with formulae.

### 2.4.1 Derivation of formulae for finding the payment amount

1. If given the future value in the problem, then we use Equation 2.2. We begin by substituting everything in, then divide both sides by the value of the accumulation factor,  $\frac{(1+i)^n - 1}{i}$ . The result will be the payment amount. Similarly one could construct a formula for the payment amount using the same idea.

$$FV_n = PMT \left[ \frac{(1+i)^n - 1}{i} \right] \quad \text{Equation 2.2}$$

$$\frac{FV_n}{\left[ \frac{(1+i)^n - 1}{i} \right]} = \frac{PMT \left[ \frac{(1+i)^n - 1}{i} \right]}{\left[ \frac{(1+i)^n - 1}{i} \right]} \quad \text{Divide both sides by accumulation factor}$$

$$\frac{FV_n}{\left[ \frac{(1+i)^n - 1}{i} \right]} = PMT \quad \text{Cancel out factors on RHS} \quad (2.7)$$

Many students will likely find it much easier just to work with Equation 2.2 and not worry about this newly derived Equation 2.7. Each student may make a choice.

2. If given the present value in the problem, then we use Equation 2.5. The process is very similar to the above processes given the future value. Substitute all information into the equation, evaluate all arithmetic, and then divide by the result of the discount factor,  $\frac{1-(1+i)^{-n}}{i}$ . We could also construct a formula using a similar process as the procedure above. We just give the formula here as the derivation is nearly the same.

$$PMT = \frac{PV_n}{\left[ \frac{1 - (1+i)^{-n}}{i} \right]} \quad (2.8)$$

### 2.4.2 Applications when the future value is given

**Example 2.4.1.** What deposit made at the end of each quarter will amount to \$15,000 in five years at 5% compounded quarterly?

**Solution:** We are looking for the payments,  $PMT$  in Equation 2.2 or 2.7. Here we will use Equation 2.2, as generally students find this easier, with  $FV_n = 15000$ ,  $i = \frac{0.05}{4} = 0.0125$ , and  $n = (4)(5) = 20$ . To solve this problem using this method we need to

compute the accumulation factor and then divide both sides by the result. So

$$\begin{aligned}
 FV_n &= PMT \left[ \frac{(1+i)^n - 1}{i} \right] \\
 15000 &= PMT \left[ \frac{(1+0.0125)^{20} - 1}{0.0125} \right] \\
 15000 &= PMT(22.5629785\dots) && \text{Computed accumulation factor on RHS} \\
 \frac{15000}{22.5629785\dots} &= PMT && \text{Divide both sides by accumulation factor} \\
 664.8058445\dots &= PMT
 \end{aligned}$$

Thus the payment made at the end of each quarter is about \$664.81. ◀

**Example 2.4.2.** Roland wants to save money for a project he wants to complete on his house. He needs \$22,000 to buy materials for the project and is guaranteed an account returning 4% compounded monthly. How much will he have to put into this account at the end of each month to generate the amount needed to purchase the materials within 7 years?

**Solution:** We are looking for the payments again,  $PMT$  in Equation 2.2 or 2.7. Again, we will use Equation 2.2. with  $FV_n = 22000$ ,  $i = \frac{0.04}{12} = 0.00\bar{3}$ , and  $n = (12)(7) = 84$ . So

$$\begin{aligned}
 FV_n &= PMT \left[ \frac{(1+i)^n - 1}{i} \right] \\
 22000 &= PMT \left[ \frac{(1+0.00\bar{3})^{84} - 1}{0.00\bar{3}} \right] \\
 22000 &= PMT(96.7541592\dots) && \text{Computed accumulation factor on RHS} \\
 \frac{22000}{96.7541592\dots} &= PMT && \text{Divide both sides by accumulation factor} \\
 227.3804061\dots &= PMT
 \end{aligned}$$

Thus the necessary payment at the end of each month is about \$227.38 to generate the \$22,000 Roland needs in 7 years. ◀

### 2.4.3 Applications when the present value is given

**Example 2.4.3.** What semi-annual payment at the end of each six-month period is required to pay off a loan of \$8000 in ten years if interest is 10% compounded semi-annually?

**Solution:** We are looking for the payments again,  $PMT$  but in this case since we are paying off a loan we will use the present value formula (Equation 2.5 or 2.8) for an ordinary simple annuity. We will use Equation 2.5, as many students find this method computationally simpler. In this equation we have  $PV_n = 8000$ ,  $i = \frac{0.10}{2} = 0.05$ , and

$n = (2)(10) = 20$ . So

$$PV_n = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \quad \text{Equation 2.5}$$

$$8000 = PMT \left[ \frac{1 - (1 + 0.05)^{-20}}{0.05} \right]$$

$$8000 = PMT(12.46221034\dots) \quad \text{Computed discount factor on RHS}$$

$$\frac{8000}{12.46221034\dots} = PMT \quad \text{Divide both sides by discount factor}$$

$$641.9406975\dots = PMT$$

Thus the necessary payment at the end of each six month period is \$641.94. ◀

**Example 2.4.4.** Dan typically has a problem paying his bills and has run out of money before his next paycheck. He makes the choice to get a payday loan from a PayDay Advance Store. As it turns out Dan borrowed \$1000 from the store and the agreement says, in fine print, that interest is 22% p.a. compounded weekly. Dan plans to pay off the loan over 1 year and 6 months. What payments must he make at the end of each week to pay off the loan? What total amount of money will Dan eventually pay back on this loan?

**Solution:** We are looking for the payments again,  $PMT$  and since we are dealing with a loan we will use the present value formula (Equation 2.5) for an ordinary simple annuity again. We will use Equation 2.5. In this equation we have  $PV_n = 1000$ ,  $i = \frac{0.22}{52} = 0.004230769\dots$ , and  $n = (52)(1.5) = 78$ . We use 1.5 years here because 6 months is half of a year. So

$$PV_n = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \quad \text{Equation 2.5}$$

$$1000 = PMT \left[ \frac{1 - (1 + 0.004230769\dots)^{-78}}{0.004230769\dots} \right]$$

$$1000 = PMT(66.31787895\dots) \quad \text{Computed discount factor on RHS}$$

$$\frac{1000}{66.31787895\dots} = PMT \quad \text{Divide both sides by discount factor}$$

$$15.07889\dots = PMT$$

Thus the necessary payment at the end of each week is \$15.08.

To answer the question as to how much Dan will eventually pay back for his PayDay Advance loan we just need to multiply the figure generated above with the total number of payments:  $15.08 \times 78 = \$1176.24$ . ◀

## 2.5 Finding the Number of Compounding Periods in an Ordinary Simple Annuity

This section corresponds to section 11.5 in the textbook and deals with finding the variable  $n$  in various problems where we are given other information. As with the previous section there are two global considerations that we must deal with: How to find  $n$  when the future value is given and how to find  $n$  when the present value is given.

In this section we will begin to need a financial calculator to compute some of the values as computation will become overly complex and in some cases near impossible without a computer, calculator, or numerically analytic method. From this point on students should be aware of the following notation,  $\boxed{\text{button}}$ . We will use this symbol to describe a special button on the Texas Instruments BA II Plus Financial Calculator. This will not necessarily represent a button on any other calculator.

### 2.5.1 Formulae for finding the number of payments

1. If the future value is given, then we can use Equation 2.2 and solve for  $n$ . We will not derive this formula here as it makes extensive use of logarithms and is not particularly insightful for the finance student.

$$n = \frac{\ln\left(\frac{iFV_n}{PMT} + 1\right)}{\ln(1 + i)} \quad (2.9)$$

As can be seen the order of operations could cause problems here for all but those that have the strongest mathematics background, thus we will discuss how to find in  $n$  with the recommended course calculator in examples. Students that wish to see this formula used in practice, please feel free to ask questions.

2. If the present value is given, then we can use Equation 2.5 and solve for  $n$ . So

$$n = -\frac{\ln\left(1 - \frac{iPV_n}{PMT}\right)}{\ln(1 + i)} \quad (2.10)$$

Again, we will discuss how to use the calculator to find this  $n$  as the arithmetic is very error prone due to a confusing order of operations. Any student that would like to know more about this formula should ask questions.

### 2.5.2 Finding the number of compounding periods when the future value is given

**Example 2.5.1.** How long will it take for \$200 deposited at the end of each quarter to amount to \$5726.70 at 6% compounded quarterly?

**Solution:** We are looking for a length of time in years here but we will first find  $n$  which is in units of quarters. First list all the known information:  $FV = 5726.70$ ; Interest rate per year =  $\boxed{\text{I/Y}} = 6\%$ ;  $PMT = \boxed{\text{PMT}} = 200$ ; we also know that interest is compounded quarterly (4 times per year) and that the payments per year match this (4 times).

To find  $n = \boxed{\text{N}}$  on the TI BAI Plus calculator. We will always have to go through a series of steps.

Step 1 Clear all data by pressing  $\boxed{2\text{ND}}$  and then  $\boxed{\text{FV}}$ .

As you can see the second function for the  $\boxed{\text{FV}}$  key is "CLR TVM", this means "Clear Time Value of Money Worksheet". From this point on students should clear the TVM worksheet at the beginning of each problem.

Step 2 Set the payments per year (P/Y) and compounding periods per year (C/Y).

To set these press  $\boxed{2\text{ND}}$  and then  $\boxed{\text{I/Y}}$ , the symbols,  $P/Y =$ , should appear in the window. Notice the second function for  $\boxed{\text{I/Y}}$  is P/Y. This means payments per year. We know those to be 4 in our problem because the account is compounded quarterly so type in 4  $\boxed{\text{ENTER}}$ , not to be confused with the  $\boxed{=}$  key. Next, press  $\boxed{\downarrow}$ . You will now see the text "C/Y=" in the window. This wants you to set the number of compounding periods per year. We already know that these are also 4 because this is an ordinary annuity, so type in 4  $\boxed{\text{ENTER}}$ .

Now press  $\boxed{2\text{ND}}$  and  $\boxed{\text{CPT}}$ . Notice the second function of  $\boxed{\text{CPT}}$  is "QUIT". This takes us out of P/Y and C/Y mode.

Step 3 Input all other variables and compute  $n$ .

Type the following sequence of keys on the calculator:

0  $\boxed{\text{PV}}$ , 5726.70  $\boxed{\text{FV}}$ , 6  $\boxed{\text{I/Y}}$ , 200  $\boxed{+|-}$   $\boxed{\text{PMT}}$ ,  $\boxed{\text{CPT}}$   $\boxed{\text{N}}$

This sequence of keys should give us an output of 23.99998534 or 24 quarters.

To convert this value to years we simply compute  $\frac{24}{4} = 6$  years. ◀

A few comments about the keying sequence and their meanings. First, we set  $0 = PV$  because we do not have a present value in the problem. Secondly, we set  $-200 = PMT$  because this particular calculator interprets  $PV$  and  $PMT$  in terms of something called *Cash Flows* or *Outlays*, which we will talk about later. Thus anytime you are inputting  $PV$  or  $PMT$  it should always be negative, further if you are using this calculator to solve for  $PV$  or  $PMT$  the answer will be given to you as a negative number. But be aware you should not record it in your answer as a negative number. Lastly, we finished with  $\boxed{\text{CPT}}$   $\boxed{\text{N}}$ , here "CPT" means compute. Thus this part of the key sequence is asking the calculator to solve for or find  $n$ .

**Example 2.5.2.** In how many payments will your bank account grow to \$3000 if you deposit \$150 at the end of each month and the account earns 9% compounded monthly?

**Solution:** We are looking for a length of time in payments in units of months here, so we are looking for  $n$ . First, list all the known information:  $FV = 3000$ ; Interest rate per year =  $I/Y = 9\%$ ;  $PMT = 150$ ; we also know that interest is compounded monthly (12 times per year) and that the payments per year match this (12 times). So  $P/Y = 12$  and  $C/Y = 12$ .

So using the calculator, begin by clearing TVM and then setting P/Y and C/Y=12. See previous problem for help and make sure to press the Enter key after assigning 12 to each. Next, substitute all variables and solve for  $n$ :

$$0 \text{ [PV]}, 3000 \text{ [FV]}, 9 \text{ [I/Y]}, 150 \text{ [+|-] [PMT]}, \text{ [CPT] [N]}$$

The output here gives us  $n = 18.7047196 \dots$  or about 19 months to accumulate according to these terms. ◀

### 2.5.3 Finding the number of compounding periods when the present value is given

In these problem, we will not be given the future value but instead will be given a present value and need to find  $n$ . As in the previous subsection we could use Equation 2.10 and students should feel free to ask questions and ask to see problems worked using this formula. However, we will use the calculator to generate our answers quickly and efficiently, without such a high risk of error.

**Example 2.5.3.** How many quarterly payments of \$600 are required to repay a loan of \$5400 at 6% compounded quarterly?

**Solution:** We are looking for a length of time in payments in units of quarters here, so we are looking for  $n$ . First, list all the known information:  $PV = 5400$  because we are looking at a loan; Interest rate per year =  $I/Y = 6\%$ ;  $PMT = 600$ ; we also know that interest is compounded quarterly (4 times per year) and that the payments per year match this (4 times). So  $P/Y = 4$  and  $C/Y = 4$ .

So using the calculator, begin by clearing TVM and then setting P/Y and C/Y=4. See previous problems for help and make sure to press the Enter key after assigning 4 to each. Next, substitute all variables and solve for  $n$ :

$$0 \text{ [FV]}, 5400 \text{ [PV]}, 6 \text{ [I/Y]}, 600 \text{ [+|-] [PMT]}, \text{ [CPT] [N]}$$

The output here gives us  $n = 9.7407178 \dots$  or about 10 quarters. ◀

**Example 2.5.4.** On his retirement, Art received a bonus of \$8000 from his employer. Taking advantage of the existing tax legislation, he invested the money in an annuity that provides for semi-annual payments of \$1200 at the end of every six months. If interest is 6.25% compounded semi-annually, how long will the annuity exist?

**Solution:** We are looking for  $n$ . First, list all the known information:  $PV = 8000$ ; Interest rate per year =  $I/Y = 6.25\%$ ;  $PMT = 1200$ ; we also know that interest is compounded semi-annually (2 times per year) and that the payments per year match this (2 times). So  $P/Y = 2$  and  $C/Y = 2$ .

So using the calculator, begin by clearing TVM and then setting P/Y and C/Y=2. See previous problems for help and make sure to press the Enter key after assigning 2 to each. Next, substitute all variables and solve for  $n$ :

$$0 \text{ [FV]}, 8000 \text{ [PV]}, 6.25 \text{ [I/Y]}, 1200 \text{ [+|-] [PMT]}, \text{ [CPT] [N]}$$

The output here gives us  $n = 7.591884\dots$  or  $\frac{7.591884\dots}{2} = 3.7959418\dots$  years. So about 4 years. ◀

## 2.6 Finding the Periodic Interest Rate of an Ordinary Simple Annuity

This section corresponds to section 11.6 in the textbook and is interested in finding the variable  $i$  in Equations 2.3 and 2.5. Unfortunately, in general,  $i$  cannot be solved for algebraically, in terms of elementary functions. We can only approximate  $i$  with numerical methods derived from calculus and other fields of mathematics. These fields of mathematics are beyond the scope of this course. However, application problems from finance appear regularly that need us to be able to find  $i$ . Conveniently, the TI BAII Plus Financial Calculator has a nice algorithm built in from calculus to give us a pretty accurate approximation of  $i$ . All financial calculators will have some algorithm built in. In fact, the process is as easy as the problems from the previous section except instead of computing  $n$  we will want to compute  $i$ . Students may notice a slight delay in the computation of  $i$  on the calculators. This is a result of the calculator using significant memory to accurately approximate the value. In some cases, computation may take several seconds!

**Example 2.6.1.** Compute the nominal annual rate of interest at which \$100 deposited at the end of each month for ten years will amount to \$15,000.

**Solution:** Here we are looking for the nominal rate of interest. Since we are using the financial calculator, it will automatically give us the nominal rate, we will not have to convert. First, list all the known information:  $FV = 15000$ ;  $n = (12)(10) = 120$ ;  $PMT = 100$ . So  $P/Y = 12$  and  $C/Y = 12$ .

So using the calculator, begin by clearing TVM and then setting P/Y and C/Y=12. See previous problems for help and make sure to press the Enter key after assigning 12 to each. Next, substitute all variables and solve for I/Y:

$$15000 \text{ [FV]}, 0 \text{ [PV]}, 120 \text{ [N]}, 100 \text{ [+|-] [PMT]}, \text{ [CPT] [I/Y]}$$

The output here gives us  $I/Y = 4.3500567\dots\%$  or a nominal interest rate of about 4.4%. ◀

**Example 2.6.2.** A loan of \$6000 is paid off over five years by monthly payments of \$120.23. What is the nominal annual rate of interest on the loan?

**Solution:** Here we are looking for the nominal rate of interest again. Since we are using the financial calculator, it will automatically give us the nominal rate, we will not have to convert. First, list all the known information:  $PV = 6000$  because we are dealing with a loan;  $n = (12)(5) = 60$ ;  $PMT = 120.23$ . So  $P/Y = 12$  and  $C/Y = 12$ .

So using the calculator, begin by clearing TVM and then setting P/Y and C/Y=12. See previous problems for help and make sure to press the Enter key after assigning 12 to each. Next, substitute all variables and compute I/Y:

0  $\boxed{\text{FV}}$ , 6000  $\boxed{\text{PV}}$ , 60  $\boxed{\text{N}}$ , 120.23  $\boxed{+|-}$   $\boxed{\text{PMT}}$ ,  $\boxed{\text{CPT}}$   $\boxed{\text{I/Y}}$

The output here gives us  $I/Y = 7.50080096\dots\%$  or a nominal interest rate of about 7.5%. ◀

## 3 Ordinary General Annuities

This chapter corresponds to Chapter 12 in the course textbook. Here we will focus on essentially the same topics as the previous chapter, section by section, except we will be looking at Ordinary General Annuities.

In this chapter we will limit formula derivation because the mathematics involved is somewhat more complex than with Ordinary Simple Annuities and will not benefit, in a significant way, the student of finance. Students that would like a deeper knowledge of the formulae here should email or discuss this with the instructor outside of class.

The major difference between this chapter and the previous chapter is the fact that we will be dealing with a general annuity versus a simple annuity. The implication of this is that the payment period may not match the interest compounding (conversion) period. This unfortunately creates more complex mathematical formulae, but for those of you that prefer to use the financial calculator, it will not create any significant difficulty.

One other source of difficulty in this chapter is the treatment of the variable  $n$  which in the past has been used to represent the number of compounding or conversion periods. In this chapter  $n$  will represent the number of periodic payments which, because we are now dealing with general annuities, is not the same as the number of compounding periods. Students will be reminded of this issue a few times during the chapter with bold print, but should be advised to be careful when  $n$  is involved.

### 3.1 Formula for the Future value of an Ordinary General Annuity

This section corresponds to section 12.1 in the textbook and is interested in how to find the future value of an ordinary general annuity. To find this we begin by letting

$$\begin{aligned}FV_g &= \text{Future value of an ordinary general annuity} \\PMT &= \text{Size of the periodic payment} \\n &= \text{Number of **periodic payments**} \\c &= \frac{\text{Number of interest compounding periods per year}}{\text{Number of payment periods per year}} \\i &= \text{Interest rate per compounding period} \\p &= \text{Equivalent rate of interest per payment period}\end{aligned}$$

then

$$FV_g = PMT \left[ \frac{(1+p)^n - 1}{p} \right] \text{ where } p = (1+i)^c - 1 \quad (3.1)$$

This formula implies a two step process for finding the future value: Step 1 will be to find  $p$ , and step 2 will be to use  $p$  to find the future value,  $FV_g$ .

### 3.1.1 Practice at finding the equivalent rate of interest per payment period

**Example 3.1.1.** Rahmean has an annuity that he will pay \$300 into at the end of each month, for the next 6 years at 5% compounded quarterly. Find the equivalent rate of interest per monthly payment.

**Solution:** Here we are being asked to find  $p$ , nothing more. To find  $p$  using the information in Equation 3.1 we need  $i$  and  $c$ . Reading the problem we know that

$$\begin{aligned}i &= \frac{r}{m} \\ &= \frac{0.05}{4} \\ &= 0.0125\end{aligned}$$

and

$$\begin{aligned}c &= \frac{\text{Number of interest compounding periods per year}}{\text{Number of payment periods per year}} \\ &= \frac{4}{12} \\ &= 0.\bar{3}\end{aligned}$$

further

$$\begin{aligned}p &= (1 + i)^c - 1 \\ &= (1 + 0.0125)^{0.\bar{3}} - 1 \\ &= 0.00414943\dots\end{aligned}$$

So the equivalent rate of interest per monthly payment period is about 0.415%. ◀

**Example 3.1.2.** TD Canada Trust pays 3.2% compounded monthly on its high end savings account. If Ned deposits \$1000 into this type of account at the end of every six months, what is the equivalent rate of interest per semi-annual payment?

**Solution:** Here we are being asked to find  $p$  again. To find  $p$  using the information in Equation 3.1 we need  $i$  and  $c$ . Reading the problem we know that

$$\begin{aligned}i &= \frac{r}{m} \\ &= \frac{0.032}{12} \\ &= 0.002\bar{6}\end{aligned}$$

and

$$\begin{aligned}c &= \frac{\text{Number of interest compounding periods per year}}{\text{Number of payment periods per year}} \\ &= \frac{12}{2} \\ &= 6\end{aligned}$$

further

$$\begin{aligned}p &= (1 + i)^c - 1 \\ &= (1 + 0.002\bar{6})^6 - 1 \\ &= 0.01610705\dots\end{aligned}$$

So the equivalent rate of interest per semi-annual payment period is about 1.61%. ◀

**Example 3.1.3.** Looking back at Example 3.2.1, how much will be in Rahmean's account at the end of the 6 year term of the annuity?

**Solution:** We have already completed some of the work by finding  $c$  in the previous example. We just need to use the future value part of Equation 3.1. Here  $PMT = 300$ ,  $n = (12)(6) = 72$ ,  $c = 0.00414943\dots$  So

$$\begin{aligned}FV_g &= PMT \left[ \frac{(1 + p)^n - 1}{p} \right] \\ &= 300 \left[ \frac{(1 + 0.00414943)^{72} - 1}{0.00414943} \right] \\ &= 25113.19783\dots\end{aligned}$$

So the future value of the account in 6 years will be \$25,113.20. ◀

**Example 3.1.4.** Looking back at Example 3.2.2, if the maturity date of Ned's account is 10 years from now, how much will the account be worth?

**Solution:** We have already completed some of the work by finding  $c$  in the previous example. We just need to use the future value part of Equation 3.1. Here  $PMT = 1000$ ,  $n = (2)(10) = 20$ ,  $c = 0.01610705\dots$  So

$$\begin{aligned}FV_g &= PMT \left[ \frac{(1 + p)^n - 1}{p} \right] \\ &= 1000 \left[ \frac{(1 + 0.01610705)^{20} - 1}{0.01610705} \right] \\ &= 23377.43005\dots\end{aligned}$$

So the future value of the account in 10 years will be about \$23,377.43. ◀

### 3.1.2 Using the financial calculator for finding the future value of an ordinary general annuity

The TI BAI Plus Financial Calculator makes finding the future value of an Ordinary General Annuity quite easy. We will walk you through a quick example of how to generate the values, but students should seriously consider going back and reworking the previous two examples using the calculator for practice. Students that choose to use the financial calculator are reminded to check your work carefully as no partial marks are given under these circumstances.

**Example 3.1.5.** Logan is saving money to buy new parts for his computer by contributing money to an annuity. He plans to contribute \$50 at the end of each week to an account returning 3% compounded quarterly. How much money will he have in 1 year and 6 months, if that is roughly when he wishes to purchase the parts?

**Solution:** Here we are looking for  $FV$ . First, list all the known information:  $N = (52)(1.5) = 78$ ,  $PMT = 50$ ,  $I/Y = 3$ ,  $P/Y = 52$  and  $C/Y = 4$ . Here we use 1.5 for the number of years because 6 months is half of a year.

So using the calculator, begin by clearing TVM and then setting  $P/Y = 52$  and  $C/Y = 4$ , remember to quit after entering these. See previous problems for help and make sure to press the Enter key after entering each. Notice that Present Value is not involved so we set it to 0. Next, substitute all variables and solve for  $FV$ :

$$3 \text{ (I/Y)}, 0 \text{ (PV)}, 78 \text{ (N)}, 50 \text{ (+|-) (PMT)}, \text{ (CPT) (FV)}$$

The output here gives us  $FV = 3987.5976\dots$  or a future value of about \$3987.60. ◀

## 3.2 Finding the Present Value of an Ordinary General Annuity

This section corresponds to section 12.2 in the course textbook and involves finding the present value of an ordinary general annuity. Examples of such situations will often involve finding the value of a loan given to someone or some organization or some similar financial instrument.

### 3.2.1 Present value of an ordinary general annuity using the equivalent rate of interest per payment period

Recalling from the previous section that we can convert a given periodic rate of interest  $i$  into an effective rate of interest per payment period using

$$p = (1 + i)^c - 1 \tag{3.2}$$

where

$$c = \frac{\text{Number of interest compounding periods per year}}{\text{Number of payment periods per year}}$$

then the present value of an ordinary general annuity is given by

$$PV_g = PMT \left[ \frac{1 - (1 + p)^{-n}}{p} \right] \quad (3.3)$$

where  $n$  is the **total number of periodic payments**.

**Example 3.2.1.** We wish to repay a loan by making payments of \$300 at the end of every 3 months over 7 years. If interest is 9% compounded semi-annually, what was the amount of the original loan?

**Solution:** Here we are looking for  $PV$  in Equation 3.2. First, list all the known information:  $n = (4)(7) = 28$ ,  $PMT = 300$ ,  $i = \frac{0.09}{2} = 0.045$ , and  $c = \frac{2}{4} = 0.5$ .

Substituting we need to find  $p$  first, thus we have

$$\begin{aligned} p &= (1 + i)^c - 1 \\ &= (1 + 0.045)^{0.5} - 1 \\ &= 0.022252415 \dots \end{aligned}$$

Now to find  $PV_g$  using Equation 3.2

$$\begin{aligned} PV_g &= PMT \left[ \frac{1 - (1 + p)^{-n}}{p} \right] \\ &= 300 \left[ \frac{1 - (1 + 0.022252415)^{-28}}{0.022252415} \right] \\ &= 6201.939937 \dots \end{aligned}$$

Thus the loan originally amounted to \$6201.94. ◀

**Example 3.2.2.** Some agreement is made to pay off a loan in payments of \$2000 every six months for 10 years at 8% compounded monthly. How much was the original loan worth?

**Solution:** Here we are looking for  $PV$  and will use the financial calculator to solve. First, list all the known information:  $n = (2)(10) = 20$ ,  $PMT = 2000$ ,  $I/Y = 8$ ,  $P/Y = 2$  and  $C/Y = 12$ .

So using the calculator, begin by clearing TVM and then setting  $P/Y = 2$  and  $C/Y = 12$ , remember to quit after entering these and make sure to press the Enter key after entering each. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for PV:

$$8 \text{ (I/Y)}, 0 \text{ (FV)}, 20 \text{ (N)}, 2000 \text{ (+/-) (PMT)}, \text{ (CPT) (PV)}$$

The output here gives us  $PV = 27019.47935$  or \$27,019.48 was the originally value of the loan. ◀

**Example 3.2.3.** Amar has taken out a mortgage on a home. He is making payments of \$2300 at the end of each month at 4% compounded daily for 25 years. What was the mortgage principal?

**Solution:** Here we are looking for  $PV$  and will use the financial calculator to solve. First, list all the known information:  $n = (12)(25) = 300$ ,  $PMT = 2300$ ,  $I/Y = 4$ ,  $P/Y = 12$  and  $C/Y = 365$ .

So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 365$ , remember to quit after entering these and make sure to press the Enter key after entering each. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for PV:

$$4 \boxed{I/Y}, 0 \boxed{FV}, 300 \boxed{N}, 2300 \boxed{+|-} \boxed{PMT}, \boxed{CPT} \boxed{PV}$$

The output here gives us  $PV = 435446.6639$  or \$435,446.66 was the principal. ◀

### 3.3 Finding the Periodic Payment of an Ordinary General Annuity

This section corresponds to section 12.3 in the textbook and is about applications where we are interested in finding the periodic payment,  $PMT$ , of an ordinary general annuity. As usual we will still need to use the two step process from the previous two sections where we first find  $p$  using the equation  $p = (1+i)^c - 1$ , then using this in some modified form of Equation 3.1 or 3.3 to find  $PMT$ . We will also discuss using the financial calculator find the periodic payment.

#### 3.3.1 Formulae for finding the periodic payment

There are two classes of problems for which we will want to be able to find the periodic payment. One where we are given the future value among other information and two, where we are given the present value and other information.

In each of these cases we can solve Equations 3.1 and 3.3 for  $PMT$  by dividing through by the bracketed fractions giving us

$$PMT = \frac{p(FV_g)}{(1+p)^n - 1} \quad (3.4)$$

and

$$PMT = \frac{p(PV_g)}{1 - (1+p)^{-n}} \quad (3.5)$$

To achieve each of these formulae we have done some further simplification. None-the-less we have both formulae for finding the periodic payment in an ordinary general annuity.

**Example 3.3.1.** Arthur has taken out a loan of \$1200 from a bank that charges 9% compounded monthly. He intends to repay the loan at the end of each week over 2 years. How much should the payments be?

**Solution:** Here we are looking for  $PMT$ . First, use Equation 3.2 to find  $p$ , then use this information in Equation 3.5 to find  $PMT$ . The loan total is giving us a present value where  $n = (52)(2) = 104$ ,  $PV_g = 1200$ ,  $i = \frac{0.09}{12} = 0.0075$ , and  $c = \frac{12}{52} = 0.23076923 \dots$

To find  $p$  first we have

$$\begin{aligned} p &= (1 + i)^c - 1 \\ &= (1 + 0.0075)^{0.23076923} - 1 \\ &= 0.001725799 \dots \end{aligned}$$

Now to find  $PV_g$  using Equation 3.5

$$\begin{aligned} PMT &= \frac{p(PV_g)}{1 - (1 + p)^{-n}} \\ &= \frac{0.001725799(1200)}{1 - (1 + 0.001725799)^{-104}} \\ &= 12.61482626 \dots \end{aligned}$$

Thus each payment will need to be about \$12.61. ◀

**Example 3.3.2.** Mohammed wishes to accumulate savings to make a purchase of \$25,000. What monthly deposit (at the end of the month) would he have to make into an account that returns 7% p.a. compounded quarterly to save this amount of money in 3 years?

**Solution:** Here we are looking for  $PMT$  and we are given future value information. Thus we will use Equations 3.2 and 3.4 to solve the problem. We have  $n = (12)(3) = 36$ ,  $FV_g = 25000$ ,  $i = \frac{0.07}{4} = 0.0175$ , and  $c = \frac{4}{12} = 0.\bar{3}$ .

To find  $p$  first we have

$$\begin{aligned} p &= (1 + i)^c - 1 \\ &= (1 + 0.0175)^{0.\bar{3}} - 1 \\ &= 0.005799633 \dots \end{aligned}$$

Now to find  $FV_g$  using Equation 3.4

$$\begin{aligned} PMT &= \frac{p(FV_g)}{(1 + p)^n - 1} \\ &= \frac{0.005799633(25000)}{(1 + 0.005799633)^{36} - 1} \\ &= 626.474431 \dots \end{aligned}$$

Thus each monthly deposit will need to be about \$626.47. ◀

Let us work one with the financial calculator.

**Example 3.3.3.** Company X is in need of replacing some old equipment in one of their factories. The replacement costs are \$115,000. The company intends to raise money by investing money at the end of every six months into an investment account that typically returns 8% compounded monthly. How much should each semi-annual deposit be into this account, if they wish to purchase the replacement equipment in exactly 5 years?

**Solution:** Here we are looking for  $PMT$  and will use the financial calculator to solve. First, list all the known information:  $n = (2)(5) = 10$ ,  $FV_g = 115000$ ,  $I/Y = 8$ ,  $P/Y = 2$  and  $C/Y = 12$ .

So using the calculator, begin by clearing TVM and then setting  $P/Y = 2$  and  $C/Y = 12$ , remember to quit after entering these and make sure to press the Enter key after entering each. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for PMT:

$$8 \text{ (I/Y)}, 0 \text{ (PV)}, 10 \text{ (N)}, 115000 \text{ (+|-) (FV)}, \text{ (CPT) (PMT)}$$

The output here gives us  $PV = 9548.622118$  or \$9548.62 is the necessary semi-annual payment. ◀

### 3.4 Finding the Number of Compounding Periods in an Ordinary General Annuity

This section corresponds to section 12.4 in the course textbook. Here we are interested in solving problems that involving solve some equation for  $n$ . In some way, all of these problems will involve finding the number of compounding periods of a financial instrument. However, more than likely these problems will ask for more, such as the length of time in years of the term of a financial instrument. In order to solve these problems analytically we will need formulae for the future value and the present value of a ordinary general annuity solved for  $n$ . Of course will also need Equation 3.2 to solve these problems.

### 3.4.1 Formula derivation

1. Formula for finding  $n$  when a future value is given.

To derive the desired formula we need to start with Equation 3.1. So

$$\begin{aligned}
 FV_g &= PMT \left[ \frac{(1+p)^n - 1}{p} \right] \\
 \frac{FV_g}{PMT} &= \frac{(1+p)^n - 1}{p} && \text{Divide both sides by } PMT \\
 \frac{p(FV_g)}{PMT} &= (1+p)^n - 1 && \text{Multiply both sides by } p \\
 \frac{p(FV_g)}{PMT} + 1 &= (1+p)^n && \text{Add 1 to both sides} \\
 \ln \left[ \frac{p(FV_g)}{PMT} + 1 \right] &= \ln(1+p)^n && \text{Take logarithms on both sides} \\
 \ln \left[ \frac{p(FV_g)}{PMT} + 1 \right] &= n \ln(1+p) && \text{Equation 1.15, property of logarithms} \\
 \frac{\ln \left[ \frac{p(FV_g)}{PMT} + 1 \right]}{\ln(1+p)} &= n && \text{Divide both sides by } \ln(1+p) \quad (3.6)
 \end{aligned}$$

2. Formula for finding  $n$  when a present value is given.

The derivation of this formula has very nearly the same steps involved in the work above, thus we will save some time and space by simply giving the formula.

$$n = - \frac{\ln \left[ 1 - \frac{p(PV_g)}{PMT} \right]}{\ln(1+p)} \quad (3.7)$$

It is significantly easier to solve these problems with a financial calculator, however, we will work one problem with the appropriate formula above.

**Example 3.4.1.** Sofia is planning her daughter's higher education fund. If she expects to have to pay \$75,000 for all tuition. What term length in years will she need on an account returning 10% compounded quarterly if she makes payments of \$250 at the end of each month?

**Solution:** Here we are looking for  $n$  and we are given future value information. Thus we will use Equations 3.2 and 3.6 to solve the problem. We have  $PMT = 250$ ,  $FV_g = 75000$ ,  $i = \frac{0.10}{4} = 0.025$ , and  $c = \frac{4}{12} = 0.\bar{3}$ .

To find  $p$  first we have

$$\begin{aligned}
 p &= (1+i)^c - 1 \\
 &= (1+0.025)^{0.\bar{3}} - 1 \\
 &= 0.008264838 \dots
 \end{aligned}$$

Now to find  $n$  using Equation 3.6

$$\begin{aligned}
 n &= \frac{\ln \left[ \frac{p(FV_g)}{PMT} + 1 \right]}{\ln(1 + p)} \\
 &= \frac{\ln \left[ \frac{0.008264838(75000)}{250} + 1 \right]}{\ln(1 + 0.008264838)} \\
 &= \frac{\ln [2.4794514 + 1]}{\ln(1.008264838)} \\
 &= \frac{\ln(3.4794514)}{\ln(1.008264838)} \\
 &= 151.4875673 \dots
 \end{aligned}$$

This comes to about 151.5 months or, converting to years, we have  $\frac{151.4875673\dots}{12} = 12.6$  years. ◀

This is certainly a feasible method for finding the term of the financial instrument, but I think we can all agree that it is somewhat time consuming and prone to errors. We will work the rest of our problems with the financial calculator. Students may certainly use these formula to solve such a problem on tests and quizzes.

**Example 3.4.2.** How many years will it take for payments of \$10 deposited into an account at the end of each day to accumulate to \$1,000,000, if the account returns 9% compounded weekly?

**Solution:** Here we are looking for  $n$ , which we will have to convert to years. We will use the financial calculator to solve. First, list all the known information:  $PMT = 10$ ,  $FV_g = 1000000$ ,  $I/Y = 9$ ,  $P/Y = 365$  and  $C/Y = 52$ .

So using the calculator, begin by clearing TVM and then setting  $P/Y = 365$  and  $C/Y = 52$ , remember to quit after entering these and make sure to press the Enter key after entering each. Notice that the Present Value is not involved so we set it to 0. Next, substitute all variables and solve for  $n$ :

$$9 \text{ (I/Y)}, 0 \text{ (PV)}, 10 \text{ (PMT)}, 1000000 \text{ (+|-) (FV)}, \text{ (CPT) (N)}$$

The output here gives us  $N = 13168.10988 \dots$  days or  $\frac{13168.10988\dots}{365} = 36.07701337$  years  $\approx 36$  years. ◀

**Example 3.4.3.** Samar sold her property valued at \$350,000 for a down payment of 15% and payments of \$3000 at the end of every month. If interest is 6% compounded annually, how long will the payments have to be made?

**Solution:** Here we are looking for  $n$  again and will convert this to units of years. We will use the financial calculator to solve. Notice we are given a present value in the form of a loan. The down payment was 15% of the value of the property, so the loan is 85% of the value of the property or \$297,000.

Next, list all the known information:  $PMT = 3000$ ,  $PV_g = 297000$ ,  $I/Y = 6$ ,  $P/Y = 12$  and  $C/Y = 1$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 1$ , remember to quit after entering these and make sure to press the Enter key after entering each. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for  $n$ :

$$6 \text{ (I/Y)}, 297000 \text{ (PV)}, 0 \text{ (FV)}, 3000 \text{ (+|-) (PMT)}, \text{ (CPT) (N)}$$

The output here gives us  $N = 135.4196421 \dots$  months or  $\frac{135.4196421\dots}{12} = 11.28497$  years  $\approx 11.3$  years. ◀

### 3.5 Finding the Periodic Interest Rate of an Ordinary General Annuity

This section corresponds to section 12.5 in the textbook and involves finding the periodic interest rate,  $i$ , of an ordinary general annuity. As with the previous chapter, where we were interested in finding the interest rate, there is no simple algebraic method for solving the necessary equations for  $i$ . We would be required to invoke calculus or other advanced mathematical tools. These tools are outside the scope of our course and may be involved in a Finance Theory course in graduate school programs. Conveniently, the TI BAI Plus Financial Calculator has the necessary built in algorithms to find  $i$ . You may notice, when using the calculator to solve these problems, that there is sometimes a delay in getting an answer. This is the result of the algorithm using significant processor power and memory in you calculator.

**Example 3.5.1.** At what nominal interest rate compounded quarterly could a loan of \$40,000 be repaid with weekly payments (at the end of each week) of \$300 over 3 years?

**Solution:** Here we are looking for  $I/Y$ . We will use the financial calculator to solve. Notice we are given a present value in the form of a loan.

Next, list all the known information:  $PMT = 300$ ,  $PV_g = 40000$ ,  $N = 52(3) = 156$ ,  $P/Y = 52$  and  $C/Y = 4$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 52$  and  $C/Y = 4$ , remember to quit after entering these and make sure to press the Enter key after entering each. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for  $I/Y$ :

$$156 \text{ (N)}, 40000 \text{ (PV)}, 0 \text{ (FV)}, 300 \text{ (+|-) (PMT)}, \text{ (CPT) (I/Y)}$$

The output here gives us  $I/Y = 10.82747169 \dots \%$  or a nominal rate of 10.8%. ◀

**Example 3.5.2.** Jane intends to accumulate \$25,000 for a project in 5 years. She expects that she will be able to deposit \$1100 into an account at the end of every 3 months. What interest rate per year will she need to get from a bank, if it will be compounded semi-annually to accumulate this money?

**Solution:** Here we are looking for  $I/Y$ . We will use the financial calculator to solve.

Next, list all the known information:  $PMT = 1100$ ,  $FV_g = 25000$ ,  $N = 4(5) = 20$ ,  $P/Y = 4$  and  $C/Y = 2$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 4$  and  $C/Y = 2$ , remember to quit after entering these and make sure to press the Enter key after entering each. Notice that the Present Value is not involved so we set it to 0. Next, substitute all variables and solve for  $I/Y$ :

20  $\boxed{N}$ , 25000  $\boxed{FV}$ , 0  $\boxed{PV}$ , 1100  $\boxed{+|-}$   $\boxed{PMT}$ ,  $\boxed{CPT}$   $\boxed{I/Y}$

The output here gives us  $I/Y = 5.331233\dots\%$  or a nominal rate of about 5.3%. ◀

This concludes the chapter on ordinary general annuities, however, we will make use of this material again later on.

## 4 Annuities Due

This chapter corresponds to chapter 13 in the course textbook. Here will only cover section 13.1 and 13.2 on annuities due. Recall that annuities due are those annuities where payments are made at the beginning of the payment period. Further, we will consider the simple case where we have payment periods and compounding periods the same; and the general case of when these dates do not match.

### 4.1 Simple Annuities Due

This section corresponds to section 13.1 in the book. We are interested in simple annuities due. Unlike previous chapters, where we broke down finding different variables into several sections, here we will discuss how to find all variables in a simple annuity due in one large section.

#### 4.1.1 Formulae involved in simple annuities due

The fundamental idea behind the theory of annuities due is dependent upon the ideas behind ordinary annuities. Recall that the only difference between an annuity due and an ordinary annuity is with an annuity due the payment is made at the beginning of the period. So effectively the only difference with respect to the formula would be an additional period of compounding. So for a simple annuity due we have a future value of

$$FV_n(\text{due}) = PMT \left[ \frac{(1+i)^n - 1}{i} \right] (1+i) \quad (4.1)$$

The  $(1+i)$  on the right hand side represents the additional compounding or conversion period. The rest of the formulae will reflect this same idea.

Secondly, we have the present value of an annuity due.

$$PV_n(\text{due}) = PMT \left[ \frac{1 - (1+i)^{-n}}{i} \right] (1+i) \quad (4.2)$$

We do not generate formulae for the payment amount on an annuity due. We suggest that students use either Equation 4.1 or 4.2, depending on whether we are given the present or future value, then solve for  $PMT$ .

From this point on we recommend learning to use your calculator to find other variables like  $n$  and  $i$ . As  $i$  cannot be solved without advanced mathematical methods and  $n$  has a complex formula as can be seen below. To generate this formula we must invoke

logarithms by solving each of Equations 4.1 and 4.2 for  $n$ . First, we have the solution to Equation 4.1 for  $n$ .

$$n = \frac{\ln \left[ \frac{iFV_n(due)}{PMT(1+i)} + 1 \right]}{\ln(1+i)} \quad (4.3)$$

Finally, we have Equation 4.2 solved for  $n$

$$n = -\frac{\ln \left[ 1 - \frac{iPV_n(due)}{PMT(1+i)} \right]}{\ln(1+i)} \quad (4.4)$$

Again, students are reminded and highly recommended to use the financial calculator to solve these problems as the formulae are more complex, time consuming, and error prone due to confusion with the order of operations. We will work all of these using the calculator for these notes. Students that would like to see the answers generated by manual computation should see the instructor outside of class.

Conveniently, the TI BA II Plus calculator has a function that allows us to compute with annuities due. This function can be activated by changing the calculator into "BGN" mode. "BGN" refers to *Beginning of the Period* versus *End of the Period*. To activate "BGN" press the following key sequence:

(2nd) (PMT)

You should see "END" appear in the screen. This indicates to us that our calculator is setup to calculate ordinary annuities. We need it in "BGN" mode to calculate annuities due. So to change modes, while "END" is still on the screen press the following sequence:

(2nd) (ENTER)

"BGN" will appear in screen in large letters and a smaller "BGN" will appear in the upper right hand corner of your screen. This verifies that we are now in the correct mode to compute with annuities due. Further, if we wish to convert back to "END" mode, for ordinary annuities we should repeat all of the steps above:

(2nd) (PMT)

then

(2nd) (ENTER)

No small text in the upper right hand corner indicates ordinary annuity mode while small "BGN" text in the upper right hand corner indicates annuities due mode. So for this chapter your calculator should be showing the small "BGN" text in the upper right hand corner.

### 4.1.2 Example problems

**Example 4.1.1.** Richard is placing \$140 into an account at the beginning of each month for the next 36 months. If the account returns 5% compounded monthly, then how much will be in the account at the end of this term?

**Solution:** Here we are looking for the future value of a simple annuity due. We will use the financial calculator to solve.

Next, list all the known information:  $PMT = 140$ ,  $N = 36$ ,  $I/Y = 5$ ,  $P/Y = 12$  and  $C/Y = 12$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 12$ , remember to quit after entering these and make sure to press the Enter key after entering each. Again, make sure your calculator is in BGN mode, as we are dealing with an annuity due. Notice that the Present Value is not involved so we set it to 0. Next, substitute all variables and solve for  $FV$ :

$$5 \text{ (I/Y)}, 0 \text{ (PV)}, 36 \text{ (N)}, 140 \text{ (+|-) (PMT)}, \text{ (CPT) (FV)}$$

The output here gives us 5448.07309.... So our future value is \$5448.07. ◀

**Example 4.1.2.** What weekly payment must be made into an account at the beginning of each week to accumulate to \$17,000 over 13 years at 6.5% compounded weekly?

**Solution:** Here we are looking for the payment of a simple annuity due. We will use the financial calculator to solve.

Next, list all the known information:  $FV = 17000$ ,  $N = 676$ ,  $I/Y = 6.5$ ,  $P/Y = 52$  and  $C/Y = 52$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 52$  and  $C/Y = 52$ , remember to quit after entering these and make sure to press the Enter key after entering each. Again, make sure your calculator is in BGN mode, as we are dealing with an annuity due. Notice that the Present Value is not involved so we set it to 0. Next, substitute all variables and solve for  $PMT$ :

$$6.25 \text{ (I/Y)}, 0 \text{ (PV)}, 676 \text{ (N)}, 17000 \text{ (+|-) (FV)}, \text{ (CPT) (PMT)}$$

The output here gives us 15.99658998.... So the payment at the beginning of each week must be \$16.00. ◀

**Example 4.1.3.** Mrs. Ali has been paying \$200 at the beginning of each month for a loan she took out several years ago. The loan builds interest at a yearly rate of 8% compounded monthly. If the loan was originally \$20,000, how long has she been depositing money into this account?

**Solution:** Here we are looking for a length of time of a simple annuity due. We will use the financial calculator to solve.

Next, list all the known information:  $PV = 20000$  (because this is a loan),  $PMT = 200$ ,  $I/Y = 8$ ,  $P/Y = 12$  and  $C/Y = 12$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 12$ , remember to quit after entering these and make sure to press the Enter key after entering each. Again, make sure your calculator is in BGN mode, as we are dealing with an annuity due. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for  $N$ :

8  $\boxed{\text{I/Y}}$ , 0  $\boxed{\text{FV}}$ , 20000  $\boxed{\text{PV}}$ , 200  $\boxed{+|-}$   $\boxed{\text{PMT}}$ ,  $\boxed{\text{CPT}}$   $\boxed{\text{N}}$

The output here gives us 163.3602567... Thus she would have to have made payments for 163.4 months or  $\frac{163.3602567}{12} = 13.61335472$  or 13.6 years. ◀

## 4.2 General Annuities Due

This section corresponds to section 13.2 in the course textbook. Here we are interested in general annuities due, meaning we will be dealing with problems where the payments are made at the beginning of the period with the additional condition that the compounding and payment periods will not match. As discussed in the previous section, the formulae involved are increasingly complicated to derive, with limited benefit in understanding for the student of finance. Thus we will simply give the formulae without formal derivation in the first subsection.

### 4.2.1 Formulae for general annuities due

As we go through these formula you will recognize expressions from both the previous section and the chapter on ordinary general annuities.

The first formula we give is the future value of an ordinary general annuity.

$$FV_g(\text{due}) = PMT \left[ \frac{(1+p)^n - 1}{p} \right] (1+p) \quad (4.5)$$

where  $p = (1+i)^c - 1$  and  $c = \frac{\text{Number of Interest Compounding Periods per Year}}{\text{Number of Payment Periods per Year}}$

Secondly, we have the formula for the present value of a general annuity due.

$$PV_g(\text{due}) = PMT \left[ \frac{1 - (1+p)^{-n}}{p} \right] (1+p) \quad (4.6)$$

where  $p$  is the same as in Equation 4.5.

To find the payment amount given either the future or present value, we would use Equations 4.5 or 4.6 respectively and solve for  $PMT$ . We do not list the formula here because it is significantly to solve these types of problems by using the above equations than to use the more complicated form that will result from solving these equations in literal form.

The next formula can be used to solve problems that ask for time with respect to general annuities due.

$$n = \frac{\ln \left[ \frac{pFV_g(\text{due})}{PMT(1+p)} + 1 \right]}{\ln(1+p)} \quad (4.7)$$

Finally, we have Equation 4.2 solved for  $n$

$$n = - \frac{\ln \left[ 1 - \frac{pPV_g(\text{due})}{PMT(1+p)} \right]}{\ln(1+p)} \quad (4.8)$$

As students can see these formulae are highly complex.

It should be expected at this point that there is no simple formula for finding the periodic rate of interest with a general annuity due. We are once again forced to rely on our calculator's pre-programming as the formulae depend on advanced applied mathematical methods.

Students are reminded and highly recommended to use the financial calculator to solve these problems as the formulae are more complex, time consuming, and error prone due to confusion with the order of operations. We will work all of these using the calculator for these notes. Students that would like to see the answers generated by manual computation should see the instructor outside of class.

### 4.2.2 Example problems

**Example 4.2.1.** Compute the nominal annual rate of interest compounded monthly at which \$500 deposited at the beginning of every three months for ten years will amount to \$30,000.

**Solution:** Here we are looking for the nominal rate of interest of a general annuity due. We will use the financial calculator to solve.

Next, list all the known information:  $PMT = 500$ ,  $N = 40$ ,  $FV = 30000$ ,  $P/Y = 4$  and  $C/Y = 12$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 4$  and  $C/Y = 12$ , remember to quit after entering these and make sure to press the Enter key after entering each. Again, make sure your calculator is in BGN mode, as we are dealing with an annuity due. Notice that the Present Value is not involved so we set it to 0. Next, substitute all variables and solve for  $I/Y$ :

$$30000 \text{ [FV]}, 0 \text{ [PV]}, 40 \text{ [N]}, 500 \text{ [+|-] [PMT]}, \text{ [CPT]} \text{ [I/Y]}$$

The output here gives us 7.4845164.... So the nominal rate of interest would need to be about  $I/Y \approx 7.48\%$ . ◀

**Example 4.2.2.** Ted wants to accumulate \$140,000 in an RRSP by making annual contributions of \$5500 at the beginning of each year. If interest on the RRSP is 11% compounded quarterly, how long will Ted have to make contributions?

**Solution:** Here we are looking for a length of time of a general annuity due. The fact that we are given the amount he wishes to accumulate implies a future value. We will use the financial calculator to solve.

Next, list all the known information:  $PMT = 5500$ ,  $N = 36$ ,  $I/Y = 11$ ,  $P/Y = 1$  and  $C/Y = 4$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 1$  and  $C/Y = 4$ , remember to quit after entering these and make sure to press the Enter key after entering each. Again, make sure your calculator is in BGN mode, as we are dealing with an annuity due. Notice that the Present Value is not involved so we set it to 0. Next, substitute all variables and solve for  $N$ :

$$11 \text{ [I/Y]}, 0 \text{ [PV]}, 140000 \text{ [FV]}, 5500 \text{ [+|-] [PMT]}, \text{ [CPT]} \text{ [N]}$$

The output here gives us 11.8491877... or about  $n \approx 12$  years. ◀

**Example 4.2.3.** Tom just finished paying off a loan that he took out 15 years ago and has been making payments of \$2100 at the beginning of each month at 4% compounded weekly. How much money did Tom initially borrow?

**Solution:** Here we are looking for the present value of a general annuity due because we are dealing with a loan again. We will use the financial calculator to solve.

Next, list all the known information:  $PMT = 2100$ ,  $N = 180$ ,  $I/Y = 4$ ,  $P/Y = 12$  and  $C/Y = 52$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 52$ , remember to quit after entering these and make sure to press the Enter key after entering each. Again, make sure your calculator is in BGN mode, as we are dealing with an annuity due. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for  $PV$ :

4  $\boxed{I/Y}$ , 0  $\boxed{FV}$ , 180  $\boxed{N}$ , 2100  $\boxed{+|-}$   $\boxed{PMT}$ ,  $\boxed{CPT}$   $\boxed{PV}$

The output here gives us  $PV = 284752.0718\dots$  So the initial loan taken out was \$284,752.07. ◀

This concludes our discussion of Annuities Due.

## 5 Amortization

This chapter corresponds to chapter 14 in the textbook and is about Amortization and Amortization Schedules. We will deal with two types of Amortizations: those involving an ordinary simple annuity and those involving an ordinary general annuity. Many of you will find most of this chapter simply a review of the materials in previous sections with the addition of dealing with amortization tables which will be automated by your calculator.

### 5.1 Amortization Involving Simple Annuities

An **amortization** is an interest bearing debit when both interest and principal are repaid by a series of equal payments made at equal intervals of time. With each amortization comes the **amortization table** which gives a detailed break down of the payment schedule, interest paid, amount paid, and balance due. As you may expect when there are many payments, (perhaps over 25 years with monthly payments comes to 300 payments), the amortization table can become quite complex and tedious. We will discuss how an amortization table is constructed in general and then the student will choose whether to construct the tables by hand or automate them by the calculator. Be aware that construction of these tables by hand will be time consuming most of the time.

Note that since amortization is a debt and hence a loan we will be using the formula for the present value of an ordinary annuity, in this section a simple annuity. Thus the formula we use is Equation 2.5.

$$PV_n = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right]$$

**Example 5.1.1.** Margo has recently purchased a home with a \$30,000 down payment and is making payments of \$2000 at the end of each month for the next 15 years. Assume that her mortgage terms include a fixed rate of 4% p.a. compounded monthly. How much was the property sold for? How much will she pay in total for the property? How much of what she paid is interest alone?

**Solution:** To figure out how much the property sold for we will need to use Equations 2.5 and 2.6. We use Equation 2.5 first with  $PMT = 2000$ ,  $i = \frac{0.04}{12} = 0.00\bar{3}$ , and

$n = 12 \times 15 = 180$  we have

$$\begin{aligned} PV_n &= PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \\ &= 2000 \left[ \frac{1 - (1 + 0.00\bar{3})^{-180}}{0.00\bar{3}} \right] \\ &= 270384.3046\dots \end{aligned}$$

Thus the loan was \$270,384.30. So using Equation 2.6 we can figure out the value of the property by adding this amount to 30000, which gives us  $270384.30 + 30000 = \$300,384.30$

The second question asks us how much total will Margo pay for the home. Well she initially paid \$30,000 and then paid \$2000 each month for 15 years that comes to  $30000 + 2000(12)(15) = \$390,000$ .

The final question asks how much of this amount was pure interest. Here we just need to take the difference of total value of the payments and the present value of the annuity. So  $360000 - 270384.30 = \$89,615.70$ . This concludes our problem. ◀

**Example 5.1.2.** [1, p. 565] A loan of \$8000 made at 6% compounded monthly is amortized over five years by making equal monthly payments.

1. What is the size of the monthly payment?
2. What is the total amount paid to amortize the loan?
3. What is the cost of financing?

**Solution:** We begin with part 1. To find the monthly payment we can simply solve Equation 2.5 for  $PMT$  or use our calculator to find the payment. We will use the calculator here. We list all the known information:  $PV = 8000$ ,  $N = 60$ ,  $I/Y = 6$ ,  $P/Y = 12$  and  $C/Y = 12$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 12$ , remember to quit after entering these and make sure to press the Enter key after entering each. **Make sure your calculator is in END mode**, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for  $PMT$ :

$$6 \text{ (I/Y)}, 0 \text{ (FV)}, 60 \text{ (N)}, 8000 \text{ (+/-) (PV)}, \text{(CPT) (PMT)}$$

The output here gives us  $154.6624122\dots$  which comes to a monthly mortgage payment of \$154.66.

Now to part 2. The total amount paid will be the total number of payments times the amount of payment. So  $154.66 \times 60 = \$9279.60$ .

Finally part 3. The cost of financing is  $9279.60 - 8000 = \$1279.60$ . ◀

**Example 5.1.3.** [1, p. 566] A loan of \$5000 made at 8% compounded quarterly is amortized over two years by making equal payments at the end of every three months.

1. What is the size of the monthly payment?

2. What is the total amount paid to amortize the loan?
3. What is the cost of financing?
4. Construct an amortization schedule of this loan.

**Solution:** We begin with part 1. To find the quarterly payment we can simply solve Equation 2.5 for  $PMT$  or use our calculator to find the payment. We will use the calculator here. We list all the known information:  $PV = 5000$ ,  $N = 8$ ,  $I/Y = 8$ ,  $P/Y = 4$  and  $C/Y = 4$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 4$  and  $C/Y = 4$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in END mode, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0. Next, substitute all variables and solve for  $PMT$ :

$$8 \text{ (I/Y)}, 0 \text{ (FV)}, 8 \text{ (N)}, 5000 \text{ (+/-) (PV)}, \text{(CPT) (PMT)}$$

The output here gives us  $682.548996\dots$  which comes to a quarterly mortgage payment of \$682.55.

Now to part 2. The total amount paid will be the total number of payments times the amount of payment. So about  $682.55 \times 8 = \$5460.40$ .

Next, part 3. The cost of financing is about  $5460.40 - 5000 = \$460.40$ .

Finally part 4, we give the amortization schedule and discuss how it is constructed after.

Payment Number	Amount Paid	Interest Paid $i = 0.02$	Principal Repaid	Outstanding Principal Balance
0				\$5000.00
1	\$682.55	\$100.00	\$582.55	4417.45
2	682.55	88.35	594.20	3823.25
3	682.55	76.47	606.08	3217.17
4	682.55	64.34	618.21	2598.96
5	682.55	51.98	630.57	1968.39
6	682.55	39.37	643.18	1325.21
7	682.55	26.50	656.05	669.16
8	682.54	13.38	669.16	0.00
TOTAL	\$5460.39	\$460.39	\$5000.00	

Table 5.1: Amortization Schedule for Example 5.1.3

We will discuss this table in pieces. Initially, the table above is constructed by making five columns as you can see above for payment number, amount paid, interest paid, etc... We will always need a "0" payment to represent the initial balance of the mortgage or loan and we will need a row to represent each of the payments  $n$ . We know from our

problem above that she will make 8 payments so 8 additional rows labeled along the left 1, 2, 3 . . . 8. We will also need a "totals" row for the purposes of checking our final work. To calculate values in the table we work from left to right starting from row 1 (payment 1). We know each payment, except perhaps the last payment will be \$682.55 from our work above. The last payment will be different because of rounding errors. So students may wish to fill in the "Amount Paid" column first with this value, except the 8th row.

Now to the computation of values within the table. Again, we have to work from left to right and top to bottom. To find the interest paid for payment 1 we simply multiply the periodic rate ( $i = 0.02$ ) times the initial balance (\$5000). This gives us \$100.00. The next column, principal repaid, is calculated by subtracting the interest paid from the amount paid in that row. The outstanding principal balance is calculated by subtracting the principal repaid from the previous payments outstanding principal balance. Thus  $5000 - 582.55 = \$4417.45$ .

Thirdly, we go to rows 2 thru 7. The steps above are repeated for each of these row.

Next, row 8 is a bit different. To calculate the amount paid in payment 8 we must first find the amount of interest due on the remaining balance (\$669.16). To figure out the interest due we multiply again by 0.02. Giving us \$13.3832. Add this back to \$669.16. Giving us 682.5432 or \$682.54. Note that the final payment amount will most likely be a bit different from all of the others. We already calculated the interest paid above. Now the principal repaid should be exactly the remaining outstanding balance from payment 7, so \$669.16. As a consequence our outstanding principal balance should be \$0.00.

Finally, the totals row is simply a sum of the values above it. We do not add up the outstanding principal balance column because this is not useful information. The total column is a useful check to make sure your work is accurate. For example, the total amount paid corresponds to part 2 above (only off by rounding error) and total interest paid corresponds to our financing cost in part 3 above (again, only off by rounding error). ◀

As you can see from the work above, constructing the amortization table is tedious, especially as you have larger numbers of payments. Conveniently, the TI BA II Plus financial calculator automates the schedule/table for you. It is highly recommended that students **study carefully** the bottom of page 567 and top of page 568 as using the calculator will save time and prevent mistakes. **Students should try to reproduce the schedule above using their calculators.**

**Example 5.1.4.** Find the annual payment on an amortized loan of \$20,000 at 5% compounded annually for 4 years. Construct an amortization schedule.

**Solution:** To find the annual payment we can simply solve Equation 2.5 for  $PMT$  or use our calculator to find the payment. We will use the calculator here. We list all the known information:  $PV = 20000$ ,  $N = 4$ ,  $I/Y = 5$ ,  $P/Y = 1$  and  $C/Y = 1$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 1$  and  $C/Y = 1$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in END mode, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0. Next, substitute

all variables and solve for  $PMT$ :

$$5 \text{ (I/Y)}, 0 \text{ (FV)}, 4 \text{ (N)}, 20000 \text{ (+|-) (PV)}, \text{ (CPT) (PMT)}$$

The output here gives us 5640.236652... which comes to an annual payment of \$5640.24. The amortization schedule is as follows.

Payment Number	Amount Paid	Interest Paid $i = 0.05$	Principal Repaid	Outstanding Principal Balance
0				\$20,000.00
1	\$5640.24	\$1000.00	\$4640.24	15,359.76
2	5640.24	767.99	4872.25	10,487.51
3	5640.24	524.38	5115.86	5371.65
4	5640.39	268.58	5371.65	\$0.00
TOTAL	\$22,560.95	\$2560.95	\$20,000	

Table 5.2: Amortization Schedule for Example 5.1.4

This concludes our problem. ◀

**Example 5.1.5.** Gurdev has taken out a mortgage of \$200,000 to purchase property at 4% compounded monthly for 15 years. If his payments are \$1479.38 at the end of each month, how much outstanding principal does he still have to pay after the 37th payment? How much interest will he pay on his 50th payment? How much of the principal has been repaid on the 100th payment? [HINT: Do not construct a table by hand.]

**Solution:** Here we already have all the information we need. We just need to input it into the calculator and generate the amortization schedule. We know that  $PV = 200000$ ,  $N = 180$ ,  $I/Y = 4$ ,  $PMT = 1479.38$ ,  $P/Y = 12$ , and  $C/Y = 12$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 12$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in END mode, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0.

$$4 \text{ (I/Y)}, 0 \text{ (FV)}, 180 \text{ (N)}, 200000 \text{ (+|-) (PV)}, 1479.38 \text{ (PMT)}, \text{ (2nd) (PV)}$$

The calculator will automatically generate the very large amortization schedule.

To answer the first question we need to look at the 37th payment. At the moment you should see P1 on the calculator screen, type 37 **(ENTER)**, then press **(↓)** to see P2 on the screen. At this point type 37 **(ENTER)** again. This isolates the information on payment 37 of the table. Pressing **(↓)** one time gives us BAL=-168,053.1863, this represents the outstanding balance on the principal. So the outstanding balance on the principal is \$168,053.19.

To answer the second question we need to look at the 50th payment. Press **(↑)** until you see P1 and repeat the steps of the previous paragraph (except use 50 instead of

37). After scrolling down you should eventually see  $\text{INT}=522.7275509$ , this represents the interest paid on the 50th payment. So the answer here is: \$522.73.

Finally, the last question is answered by looking at the 100th payment. So repeat the steps above to see the details on the 100th payment. Scroll until you see  $\text{PRN}=1,129.83814$ , this represents the amount of the principal repaid on the 100th payment. So the answer is \$1,129.84. ◀

## 5.2 Amortization Involving General Annuities

This section corresponds to section 14.2 and is basically the same as section 14.1 with the exception that the compounding period does not match the payment period, thus an ordinary general annuity is involved. We will need to invoke Equations 3.2 and 3.3. As a reminder we have

$$p = (1 + i)^c - 1$$

where

$$c = \frac{\text{Number of interest compounding periods per year}}{\text{Number of payment periods per year}}$$

then the present value of an ordinary general annuity is given by

$$PV_g = PMT \left[ \frac{1 - (1 + p)^{-n}}{p} \right]$$

The only major difference we encounter in this section, compared to the last, is in construction of the amortization schedule. We will use the equivalent rate of interest  $p$  instead of  $i$  in our schedules. However, as you may have guessed your financial calculator will do most of the work for you.

**Example 5.2.1.** [1, p. 583] A debt of \$30,000 with interest at 9.75% compounded quarterly is to be repaid by equal payments at the end of each year for seven years.

1. Compute the size of the yearly payments.
2. Construct an amortization schedule.

**Solution:** Here we are dealing with amortization and finding the payment size so we use Equations 3.2 and 3.3 or use our calculators. We use the calculator here. So we have  $PV = 30000$ ,  $N = 7$ ,  $I/Y = 9.75$ ,  $P/Y = 1$ , and  $C/Y = 4$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 1$  and  $C/Y = 4$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in END mode, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0.

$$9.75 \text{ (I/Y)}, 0 \text{ (FV)}, 7 \text{ (N)}, 30000 \text{ (+|-) (PV)}, \text{ and (CPT), (PMT)},$$

We get a result of  $PMT = 6184.969074\dots$  or \$6184.97. This concludes part 1.

We construct the amortization table below. Remember to use  $p$  in the construction of the table instead of  $i$ . Thus we need to compute  $p$ . Recall that

$$\begin{aligned} p &= (1 + i)^c - 1 \\ &= \left(1 + \frac{0.0975}{4}\right)^{\frac{4}{1}} - 1 \\ &= 1.024375^4 - 1 \\ &= 0.101123125\dots \\ &\approx 0.101123 \end{aligned}$$

Now to use this in construction of the table.

Payment Number	Amount Paid	Interest Paid $p = 0.101123$	Principal Repaid	Outstanding Principal Balance
0				\$30,000.00
1	\$6184.97	\$3033.69	\$3151.28	26,848.72
2	6184.97	2715.03	3469.94	23,378.78
3	6184.97	2364.14	3820.83	19,557.95
4	6184.97	1977.76	4207.21	15,350.74
5	6184.97	1552.31	4632.66	10,718.08
6	6184.97	1083.85	5101.12	5616.96
7	6184.96	568.00	5616.96	\$0.00
TOTAL	\$43,294.78	\$13,297.78	\$30,000.00	

Table 5.3: Amortization Schedule for Example 5.2.1

Concluding part 2 and our problem. ◀

**Example 5.2.2.** [1, p. 585] For the previous example, compute the outstanding balance after three payments.

**Solution:** There are three ways to work this problem. We will outline each of the three methods and show how to use the calculator to generate the solution.

Method 1 involves using the future value of an ordinary general annuity (Equation 3.1) and the fact that

$$\begin{aligned} \text{Outstanding Balance} &= \text{Accumulated Value of the Original Debt} \\ &\quad - \text{Accumulated Value of the Payments Made} \end{aligned}$$

Students that would like to see this method further should look at example 14.2b on page 585 of the course textbook or see the instructor outside of class.

The second method involves the programmable financial calculator. Here we will construct the amortization table. So we have  $PV = 30000$ ,  $N = 7$ ,  $I/Y = 9.75$ ,

$P/Y = 1$ ,  $PMT = 6184.97$  and  $C/Y = 4$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 1$  and  $C/Y = 4$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in END mode, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0.

9.75  $\boxed{\text{I/Y}}$ , 0  $\boxed{\text{FV}}$ , 7  $\boxed{\text{N}}$ , 30000  $\boxed{+|-}$   $\boxed{\text{PV}}$ , 6184.97  $\boxed{\text{PMT}}$ , and  $\boxed{2\text{nd}}$   $\boxed{\text{PV}}$

The calculator will automatically generate the amortization schedule seen in the previous example.

To answer the question we need to look at the 3rd payment. At the moment you should see P1 on the calculator screen, type 3  $\boxed{\text{ENTER}}$ , then press  $\boxed{\downarrow}$  to see P2 on the screen. At this point type 3  $\boxed{\text{ENTER}}$  again. This isolates the information on payment 3 of the table. Pressing  $\boxed{\downarrow}$  one time gives us BAL=-19,557.94599... So the outstanding balance is \$19,557.95.

The third method is the easiest and involves looking at the constructed schedule. Note that the table contains rounding errors that students must be aware. To find the outstanding balance after three payments look at payment 3 on the schedule in the last problem. The outstanding balance is \$19,557.95. In this case there is no error. ◀

Let us take a look at one last problem which will seem a bit challenging.

**Example 5.2.3.** Ed has taken out a mortgage for a new property of \$400,000 to be paid over 25 years with payments made every month at 3.8% compounded semi-annually. The loan can be renewed after 5 years at a rate of 4.5% compounded semi-annually.

1. What are the payment amounts for the first five years?
2. What is the balance on the loan at the end of 5 years?
3. What will the new payment amounts be when the new terms begin after 5 years?

**Solution:** As usual we will work this problem using the calculator's amortization function because the mathematics involved, however doable, are also complex.

In part 1, we are interested in the payment amounts involved with the term for the first five years. We will still assume that the term length of the loan is 25 years or  $n = 300$ . Take caution here. The issue here with five years is simply a change in terms and has little numerical implication other than that. So we have  $PV = 400000$ ,  $N = 300$ ,  $I/Y = 3.8$ ,  $P/Y = 12$ , and  $C/Y = 2$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 2$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in END mode, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0.

3.8  $\boxed{\text{I/Y}}$ , 0  $\boxed{\text{FV}}$ , 300  $\boxed{\text{N}}$ , 400000  $\boxed{+|-}$   $\boxed{\text{PV}}$ , and  $\boxed{\text{CPT}}$   $\boxed{\text{PMT}}$

The output here is  $PMT = 2060.938542 \dots$  which is \$2060.94, our answer to part 1.

To answer part 2, we must construct the amortization schedule in the calculator and look at the payment at the end of 5 years (which is the 60th payment). It is not necessary to re-input the data above with the exception of the payment. So press 2060.94  $\boxed{\text{PMT}}$ , then  $\boxed{2\text{nd}} \boxed{\text{PV}}$ . This will generate the schedule for us. With P1 on the screen press 60  $\boxed{\text{ENTER}}$ . Also set P2 to 60. This isolates the 60 payment or the payment at the end of 5 years. Scroll down to the balance. You should see  $BAL = -346,992.3805 \dots$ . Thus the remaining debt on the loan after 5 years is \$346,992.38, answering question 2.

For the final question, we must find the payment amount for the rest of the term of the loan. Note that 5 years have lapsed and thus we only have 20 years on the term now or  $n = 240$  months. The present value must also be changed to match the new balance recorded in part 2 above. Also the interest rate has changed to 4.5%. So using the calculator we have  $PV = 346992.38$ ,  $N = 240$ ,  $I/Y = 4.5$ ,  $P/Y = 12$ , and  $C/Y = 2$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 12$  and  $C/Y = 2$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in END mode, as we are dealing with an ordinary annuity. Notice that the Future Value is not involved so we set it to 0.

4.5  $\boxed{\text{I/Y}}$ , 0  $\boxed{\text{FV}}$ , 240  $\boxed{\text{N}}$ , 346992.38  $\boxed{+|-} \boxed{\text{PV}}$ , and  $\boxed{\text{CPT}} \boxed{\text{PMT}}$

The output here is  $PMT = 2187.457853 \dots$  which is \$2187.46, our answer to part 3. ◀

This concludes the section and our discussion on Amortization.

## 6 Investment Decision Applications

This chapter corresponds to chapter 16 in the course textbook. Here we will be looking a variety of application problems that deal with Investment. These applications will make wide use of many of the formulae and materials from earlier in the course.

In the following two sections we will be making use of some terminology indigenous to the investment and financial communities. It will be important that you understand what they mean. It is fairly intuitive what each will mean however we give a semi-formal definition for clarification. **Cash inflows** will be used to refer to money coming into a business or financial institution. **Cash outflow** or **Outlays** refers to money being spent, or going out of the business or institution. [1, p. 678] A **Capital Investment Project** is a project that is expected to generate benefits over a period of time longer than one year. Capital investment projects result in **Capital Expenditures**.

Obviously, there is application for making decisions about which of a set up of options, is the better in terms of capital investment projects or other financial instruments. As it turns out there are many possible schemes for which an individual or institution can use to make these decisions. Here we will have a look at two of these methodologies: Discounted Cash Flow and Net Present Value. In each case, the primary concern will be the time value of money issues for which we have been studying for the past months.

### 6.1 Discounted Cash Flow

As mentioned previously discounted cash flow is a method for making decisions about investment options. [1, pg. 679] Discounted cash flow is the present value of all cash payments. There are two considerations when using this method to make decisions:

1. The cash-in-hand principal. Given that all other factors are equal, earlier benefits are preferable to later benefits.
2. The bigger-the-better principal. Given that all other factors are equal, bigger benefits are preferable to smaller benefits.

This scheme should be used throughout all problems in this section.

**Example 6.1.1.** You are offered \$50 today or \$50 a month from now. Which is the preferred choice?

**Solution:** This problem is application of the first principal of discounted cash flow. We prefer earlier benefits to later benefits, thus \$50 today is the preferred choice. ◀

**Example 6.1.2.** You are offered \$3000 today or \$4000 five years from now. If money is worth 6%, which option is preferable?

**Solution:** First, option 1, \$3000 today has exactly that present value, \$3000. In option 2 we will have to compute the present value of \$4000. We can use Equation 1.12 or the pre-programmed financial calculator. Here we will use the formula with  $S = 4000$ ,  $i = 0.06$ , and  $n = 5$ .

$$\begin{aligned} P &= \frac{S}{(1+i)^n} \\ &= \frac{4000}{(1+0.06)^5} \\ &= 2989.032691\dots \end{aligned}$$

This gives us a present value on the \$4000 of \$2989.03 which is less than \$3000. Thus we prefer the first option and choose to take \$3000 today. ◀

In relation to the previous problem, consider if we had not been given an interest rate or value of money. In this case we would be unable to make a decision using discounted cash flow, because we would be unable to find the present value of the \$4000. See example 16.1b on page 679 of the course textbook for more details.

**Example 6.1.3.** Cyberdyne Research has been given two alternatives for purchasing a new electron microscope. Alternative 1: Purchase the microscope with cash up front. The up front cost is \$200,000. Alternative 2: Pay \$20,000 up front, then make payments at the end of each month of \$5000 for 3 years at an interest rate of 8% compounded monthly. Which alternative is best by the discounted cash flow method?

**Solution:** First, Alternative 1 has a present value of exactly \$200,000 as the money would be paid up front.

Alternative 2 is partially an ordinary simple annuity. So we need to find it's present value using Equation 2.5 or the financial calculator. We will use the equation in this instance with  $PMT = 5000$ ,  $i = 0.00\bar{6}$ , and  $n = 36$ .

$$\begin{aligned} PV_n &= PMT \left[ \frac{1 - (1+i)^{-n}}{i} \right] \\ &= 5000 \left[ \frac{1 - (1+0.00\bar{6})^{-36}}{0.00\bar{6}} \right] \\ &= 159,559.0276\dots \end{aligned}$$

This gives us a present value on the \$159,559.03. We also need to consider that we paid \$20,000 up front, so we need to add these two together giving us  $159559.0276 + 20000 = 179559.0276$  or a total present value for Alternative 2 of \$179,559.03. Thus we prefer the Alternative 2 because it will cost less with respect to the time value of money. ◀

The next example is somewhat challenging and will draw on several topics we have discussed throughout the course.

**Example 6.1.4.** A company that builds very tall buildings needs a new device that makes the foundation of buildings more able to withstand earthquakes. They cannot

purchase the device up front as they currently have no contract. The corporation that manufactures the device has given two options to purchase the device: Option 1-Pay \$150,000 up front and \$220,000 4 years from now. Option2 - Pay \$40,000 up front and pay \$50,000 at the beginning of every 6 months for the next 3 years. Assuming that all interest is 7% compounded quarterly, which option is better for the company?

**Solution:** First, Option 1 has a present value of exactly \$150,000 plus the present value of \$220,000, 4 years from now. To find the present value we need Equation 1.12 or the calculator. Here we will use the calculator with  $FV = 220000$ ,  $I/Y = 7\%$ , and  $n = 16$ . So using the calculator, begin by clearing TVM and then setting  $P/Y = 4$  and  $C/Y = 4$ , remember to quit after entering these and make sure to press the Enter key after entering each.

7  $\boxed{I/Y}$ , 16  $\boxed{N}$ , 220000  $\boxed{+|-}$   $\boxed{FV}$ , and  $\boxed{CPT}$   $\boxed{PV}$

The output here is  $PV = 166675.5874\dots$ . We need to add this to \$150,000 to get the total present value for option 1, which is \$316,675.59

Option 2 is partially a general annuity due. We will need to find the present value of payments of \$50,000 for the next 3 years and add \$40,000 to this. We can find the present value using Equation 4.6 or the financial calculator. Again, we use the calculator with  $PMT = 50000$ ,  $I/Y = 7$ , and  $N = 6$ . Remember,  $N$  in any annuity formula is referring to the payments and not the compounding periods. Begin by clearing TVM and then setting  $P/Y = 2$  and  $C/Y = 4$ , remember to quit after entering these and make sure to press the Enter key after entering each. Make sure your calculator is in BGN mode, as we are dealing with an annuity due. Notice that the Future Value is not involved so we set it to 0.

7  $\boxed{I/Y}$ , 0  $\boxed{FV}$ , 6  $\boxed{N}$ , 50000  $\boxed{+|-}$   $\boxed{PMT}$ , and  $\boxed{CPT}$   $\boxed{PV}$

The output here is  $PV = 275556.9496\dots$  which needs to be added to the original \$40,000 up front gives us \$315,556.95 as a total present value of option 2.

The cheaper option with respect to the time value of money right now is option 2 (notice by only a small amount). ◀

## 6.2 Net Present Value Method

This section corresponds to section 16.2 in the course textbook and is another method for dealing with investment decisions that builds on discounted cash flow. In this section, as well as the last, our central interest is the present value of money and using this to decide when a particular investment option is viable or which of two or more options is better.

### 6.2.1 Basic ideas

The **Net Present Value** (NPV) is the result of subtracting total outlays from total inflows. Thus

$$NPV = PV_{\text{in}} - PV_{\text{out}} \quad (6.1)$$

Clearly, there are three possible outcomes for the NPV in this case.

1.  $NPV > 0 \Rightarrow PV_{in} > PV_{out}$
2.  $NPV < 0 \Rightarrow PV_{in} < PV_{out}$
3.  $NPV = 0 \Rightarrow PV_{in} = PV_{out}$

In our problems we will be given the institution or organization's required rate of return. According to the net present value method we accept the terms when outcomes 1 or 3 occur and reject if outcome 2 occurs. The reason should be obvious: As a business owner or corporate share holder we always want our business to make money above a certain critical value. That critical value is the organization's required rate of return. As a consequence Present value of all inflows should be greater than or the same as the present value of all outlays which implies that  $NPV \geq 0$ , i.e. outcomes 1 and 3.

As you can see we will have to compute many present values. As a result we can either do the calculations by hand using appropriate formulae throughout the course or we can use the time value of money functions on our financial calculators. As it turns out, there is yet another calculator option. The TI BA II Plus has an NPV function. The process for inputting values is somewhat different from using the TVM functions on the calculator. Students are welcomed to use the NPV function and can read about how to use it between pages 688 and 694 of the course textbook, however, we will not use the NPV function officially in class to solve problems. Students that are interested in the NPV function should ask questions before class, after class, or send an email with concerns. There is no fast way to work any of these problems.

**Example 6.2.1.** Corporation X is responsible for building a certain type of product. There are several methods of making this product. One method that is being considering expects to have to pay \$50,000 immediately and \$65,000 three years from now. The product can be sold at the end of 5 years for a total of \$300,000. If Corporation X requires a 12% per year return on its investment, should this method of construction be considered?

**Solution:** To answer the question we must break this problem down into its inflows and outlays. The inflow is the \$300,000 that the company will make in 5 years for selling the product. The outlays are the \$123,000 up front and \$65,000 in three years. We must first find the present value of each of these with respect to the time value of money.

The present value of the inflows can be found using the calculator with  $FV = 300000$ ,  $I/Y = 12$ ,  $N = 5$ . Make sure that P/Y and C/Y are both set to 1. So to calculate we type

$$12 \left[ \frac{I}{Y} \right], 5 \left[ N \right], 300000 \left[ +|- \right] \left[ FV \right], \text{ and } \left[ CPT \right] \left[ PV \right]$$

The output here is  $PV = 170228.0567\dots$ . So the present value is \$170,228.06. This is  $PV_{in}$  in Equation 6.1 because there are no other inflows.

Now to the present value of the outlays. The present value of \$123,000 immediately is \$123,000. So no work there. But we need to find the present value of \$65,000 three years

from now. We will again use the TVM function on our calculators. Let  $FV = 65000$ ,  $I/Y = 12$ ,  $N = 3$ . Make sure that  $P/Y$  and  $C/Y$  are both set to 1. So to calculate we type

$$12 \text{ (I/Y)}, 3 \text{ (N)}, 65000 \text{ (+|-) (FV)}, \text{ and } \text{(CPT) (PV)}$$

The output here is  $PV = 46265.71611\dots$ . So the present value is \$46,265.72. We need to add this to the immediate outlay of \$123,000, giving us \$169,265.72. This is  $PV_{\text{out}}$  in Equation 6.1.

So we have

$$\begin{aligned} NPV &= PV_{\text{in}} - PV_{\text{out}} \\ &= 170228.06 - 169265.72 \\ &= 962.34 \\ &\geq 0 \end{aligned}$$

This implies that by the net present value method Corporation X should consider this method as viable for building their product and receive at least 12% return on investment. ◀

Our final problem for this section is lengthy and somewhat challenging.

**Example 6.2.2.** The federal government is deciding between two methods for the construction of a nuclear power station. The Type 1 Power Station involves an initial outlay of \$70 million and requires payments of \$20 million at the end of each year for 10 years. The Type 1 Station will bring an income of \$80 million per year for the next 20 years. The Type 2 Power Station involves purchasing a device used to build the station faster. This device will require an initial payment of \$120 million, but will have salvage value of \$25 million after 7, and will also require a payment of \$230 million after 3 years. The Type 2 Station will generate more power and last longer so its income will be higher at \$85 million per year for the next 25 years. Which station is better if the government wishes to get at least 10% return on its investment?

**Solution:** We will look at the Type 1 Station first and begin with the inflow from the income. We need to find the present value of \$80 million for 20 years, which is an ordinary simple annuity. So using the calculator with  $P/Y = C/Y = 1$ ,  $PMT = 80$ ,  $I/Y = 10$ , and  $N = 20$  we have

$$10 \text{ (I/Y)}, 20 \text{ (N)}, 80 \text{ (+|-) (PMT)}, \text{ and } \text{(CPT) (PV)}$$

The output here is  $PV = 681.0850976\dots$  or about \$681 million. This represents the  $PV_{\text{in}}$  for Station 1.

We now look at the outlays for Station 1. We need to find the present value of \$20 million for 10 years, which is an ordinary simple annuity. So using the calculator with  $P/Y = C/Y = 1$ ,  $PMT = 20$ ,  $I/Y = 10$ , and  $N = 10$  we have

$$10 \text{ (I/Y)}, 10 \text{ (N)}, 20 \text{ (+|-) (PMT)}, \text{ and } \text{(CPT) (PV)}$$

The output here is  $PV = 122.8913421 \dots$  or about \$123 million. This needs to be added to the initial outlay of \$70 million giving us  $PV_{\text{out}} = \$193$  million. So we have

$$\begin{aligned} NPV(\text{type 1 reactor}) &= PV_{\text{in}} - PV_{\text{out}} \\ &= 681 - 193 \\ &= 488 \\ &\geq 0 \end{aligned}$$

making Station 1 a viable option for construction.

We will now look at the Type 2 Station and begin with the inflow from the income. We need to find the present value of \$85 million for 25 years, which is an ordinary simple annuity. So using the calculator with  $P/Y = C/Y = 1$ ,  $PMT = 85$ ,  $I/Y = 10$ , and  $N = 25$  we have

$$10 \text{ (I/Y)}, 25 \text{ (N)}, 85 \text{ (+|-) (PMT)}, \text{ and (CPT) (PV)}$$

The output here is  $PV = 771.5484015 \dots$  or about \$772 million. We also will eventually have income from the salvage of the device purchased so we need to find its present value and add these present values. So we need the present value of a future value of \$25 million, 7 years earlier. So using the calculator with  $P/Y = C/Y = 1$ ,  $FV = 25$ ,  $I/Y = 10$ , and  $N = 7$  we have

$$10 \text{ (I/Y)}, 7 \text{ (N)}, 25 \text{ (+|-) (FV)}, \text{ and (CPT) (PV)}$$

The output here is  $PV = 12.828952 \dots$  or about \$13 million. Adding this with the present value of the income gives us  $PV_{\text{in}} = \$785$  million for Station 2.

We now look at the outlays for Station 2. We need to find the present value of \$230 million in 3 years. Note that this is not an annuity because there is no payment. So using the calculator with  $P/Y = C/Y = 1$ ,  $FV = 230$ ,  $I/Y = 10$ , and  $N = 10$  we have

$$10 \text{ (I/Y)}, 3 \text{ (N)}, 230 \text{ (+|-) (FV)}, \text{ and (CPT) (PV)}$$

The output here is  $PV = 172.80240 \dots$  or about \$172 million. This needs to be added to the initial outlay of \$120 million giving us  $PV_{\text{out}} = \$293$  million. So we have

$$\begin{aligned} NPV(\text{type 2 reactor}) &= PV_{\text{in}} - PV_{\text{out}} \\ &= 785 - 293 \\ &= 492 \\ &\geq 0 \end{aligned}$$

which indicates that this station is a viable construction option for the government as well. However, since the NPV of the Type 2 Station is higher, the government should build this type of reactor to get more return by the net present value method. ◀

# 7 Cost Volume Profit Analysis

## 7.1 Applications-Cost-Volume-Profit Analysis

Students should be aware that this section corresponds to section 6.1 in the textbook and in the course outline. Students should also be aware that at this point the course content changes dramatically from the previous materials. We turn our attention away from Time-Value of Money applications from the first 2/3 of the course and look at Break-Even analysis and Cost-Volume Profit Analysis. If you took BMAT110 in a previous semester you will have some familiarity with this already.

Some students may need to review sections 4.1-4.3 of the course text book as the materials in this chapter depend greatly on one's ability to work with linear equations and recognize their geometry.

**Definition 1.** *Let  $R(x)$  and  $C(x)$  be functions that describe the Total Revenue and Total Cost, respectively, of some business venture. Any solutions to the system of equations generated by these two functions is called a **Break-Even point**. The process of finding the break-even point and analyzing the geometry of the two functions  $R(x)$  and  $C(x)$  and how they relate to each other is called **Cost-Volume-Profit Analysis**.*

The textbook will use different notations for the functions (linear) discussed in the above definition. Instead of  $R(x)$  we will use  $TR$  for total revenue, not to be confused with  $T \times R$ . Instead of  $C(x)$  we will use  $TC$  for total cost.

### 7.1.1 Basic Break-Even Analysis Problems

**Example 7.1.1.** Rahmeen has had a computer repair business opened for a few months and has recorded some data relating to the income it makes and how much it costs to run. As far as costs are concerned he has to pay rent to the building owner and to an equipment company for using the space and rental of tools/equipment which comes to \$900 per month and he has estimated that he has expenses associated with supplies and employee wages of \$55 per computer repaired. As far as revenue, he charges, on average, \$75 to repair each computer. Rahmeen has also determined that, at the very most, his company could repair 100 computers in one month. How many computers would Rahmeen's business have to service in order to break-even on costs and revenues, given these estimations? The table below is a summary of some of the data above.

**Solution:** Our goal is to describe the data above in terms of a Total Cost function ( $TC$ ) and a Total Revenue function ( $TR$ ) and then to find where they are the same (find the intersection point of the two lines).

Computers Repaired	0	10	20	30	40	50	60
Revenue	\$0	\$750	\$1500	\$2250	\$3000	\$3750	\$4500
Supplies/Materials	0	550	1100	1650	2200	2750	3300
Rent Cost	900	900	900	900	900	900	900
Total Costs	900	1450	2000	2550	3100	3650	4200
Net Income	-900	-700	-500	-300	-100	100	300

Table 7.1: Rahmeen's Summary for Up to 60 Computers Repaired in One Month

First, let us consider  $TR$ , the total revenue. The company's only revenue comes from the charges for repairing computers. He gets, on average, \$75 per computer. So, if we let  $x$  be the number of computers repaired, then

$$TR = 75x \quad (7.1)$$

Next, let us look at  $TC$ , the total cost. There are two parts to the total cost of a business, variable cost  $VC$  and fixed cost  $FC$ . In other words,

$$TC = VC + FC \quad (7.2)$$

In our problem the variable cost is the costs that depend on the number of computers repaired, which is supplies/materials/employees, which costs, on average, \$55 per computer repair. So  $VC = 55x$ . His fixed costs are those costs that do not change, which is rent. So  $FC = 900$ . In summary

$$TC = 55x + 900 \quad (7.3)$$

Thus we have our system of equations,  $TR$  and  $TC$

$$\begin{cases} TR = 75x \\ TC = 55x + 900 \end{cases}$$

As an aside, we can also interpret this in the context of the previous section because we are trying to find the point where  $TR = TC$

$$\begin{cases} y = 75x \\ y = 55x + 900 \end{cases}$$

Either of these interpretations may have value for our purposes. So to solve the system we let

$$TR = TC$$

$$75x = 55x + 900$$

$$20x = 900$$

Subtract  $55x$  from both sides

$$\frac{20x}{20} = \frac{900}{20}$$

Divide both sides by 20

$$x = 45$$

This is the  $x$  value of the intersection point

We also need to find the  $y$  value of the intersection point. To do this we can substitute the  $x$  value back into either of the equations 4.1 or 4.3 above. We choose 4.1 here.

$$\begin{aligned}
 TR &= 75x \\
 &= 75(45) && \text{From the work above} \\
 &= 3375 && \text{This is the } y \text{ value of the intersection point}
 \end{aligned}$$

This number can be interpreted as the cost and revenue at the break-even point.

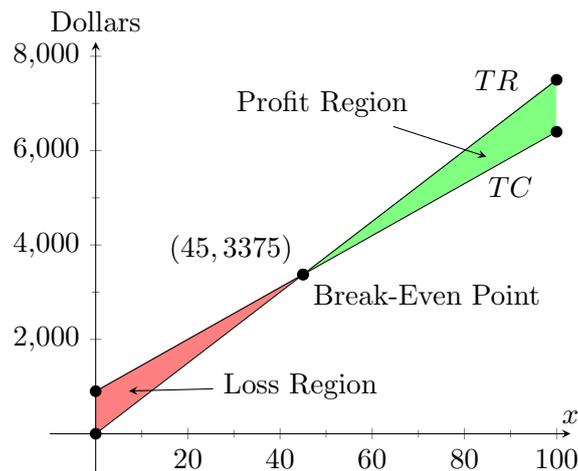
We will now discuss and draw a picture tying all of this together. We will call this picture a **Break-Even Chart**. To draw our chart we will need to know a few points on each line (equations 4.1 and 4.3). We already know a few obvious points. For example, the revenue function has a point at  $(0, 0)$ , this is its  $y$ -intercept (see slope intercept form). We also know that the break-even point is on this graph,  $(45, 3375)$ . The cost function also has a few obvious points. Its  $y$ -intercept is at  $(0, 900)$  and the break-even point is also on this line.

There are a pair of less obvious, but important points that we should know before drawing our graph. We should always know the **cost and revenue at capacity**. This is found by letting  $x$  in each function be its maximum practical or possible value. For our example we are told that Rahmeen has determined that his company can repair at most 100 computers in a month, so let  $x = 100$  and substitute into each of  $TR$  and  $TC$ .

$$\begin{aligned}
 TR &= 75(100) = 7500 \\
 TC &= 55(100) + 900 = 6400
 \end{aligned}$$

Thus we have two more points to include.  $(100, 7500)$  on the line for  $TR$  and  $(100, 6400)$  on the line for  $TC$ . Now let us summarize before we draw our chart.

	Points to be Graphed
TR	$(0,0)$ $(45,3375)$ $(100,7500)$
TC	$(0,900)$ $(45,3375)$ $(100,6400)$



Notice here that we have shaded certain parts of the graph. The shaded region to the left of the break-even point is called the **Loss Region** because here costs are more than revenues. Conversely, the region to the right of the break-even point is called the **Profit Region** because revenue is more than costs. This concludes our first Cost-Volume Profit Analysis problem. ◀

### Steps for Finding the Break-Even Point

- Step 1 Read the problem very carefully, perhaps two or more times, before doing any writing.
- Step 2 Find  $TR$ , there is only one component to  $TR$ , and it will usually depend on  $x$ . Look for words like: income or revenue. In the problem above, Rahmeen was charging his customers on average \$75 per computer repair. This is giving us information about  $TR$ .
- Step 3 Find  $TC$ . Remember  $TC = VC + FC$ , in other words there are usually two components to total cost.  $VC$  will always depend on  $x$ ,  $FC$  will be constant.
- Step 4 Solve the system of equations resulting from  $TR$  and  $TC$ , by setting  $TR = TC$ , to find the Break-Even point. Remember to find both the  $x$  and the  $y$  values.

### Steps for Drawing the Break-Even Chart

- Step 1 Find the  $y$ -intercepts of both  $TR$  and  $TC$  (see slope-intercept form). Make sure that you are clear as to what point is on which line.
- Step 2 Find the cost and revenue at capacity. Often this can be found from an  $x$  value (at capacity), given to you in the problem, that you can substitute into  $TR$  and  $TC$  for another point on each of these lines.
- Step 3 Draw the graph with just the scales on the axes. Your scales do not have to be the same on both axes, in fact most of the time they will be different. The maximum value on your  $x$  axis will need to be at or a little more than the  $x$  value given at capacity. The maximum value on your  $y$  axis will need to be at or a little more than the  $y$  value at capacity on  $TR$  (see step 2). Place intermediate ticks on the scale all the way back to zero and label them uniformly.
- Step 4 Plot your  $TR$  points generated from the steps above and remember, the break-even point from the previous step-by-step process is a point on both  $TR$  and  $TC$ . Draw a line to connect these points on  $TR$ .
- Step 5 Plot your  $TC$  points from above. Draw a line to connect these points. By now you should see that your  $TR$  and  $TC$  cross over each other at the break-even point.
- Step 6 Label the break-even point (with coordinates) and the loss and profit regions.

**Example 7.1.2.** Stephanie is a factory manager for a corporation that makes products. Stephanie has determined that the corporation incurs fixed costs of \$200 per week on property taxes and various upkeep. The corporation also has to pay the employees on average \$980 per employee per week by union contract. After discussing things with her CFO, she has determined that each employee is contributing about \$1000 to corporate revenue, further she knows that due to the small size of the factory she can have at most 20 workers on any given week. What is the minimum number of employees per week needed to keep this corporation profitable? Express the number of employees at the break-even point as a percent of capacity.

**Solution:** After reading the problem carefully we need to construct  $TR$ . We know from the problem that each employee is contributing revenue to the corporation to the tune of \$1000 per employee. So if we let  $x$  be the number of employees, then

$$TR = 1000x \quad (7.4)$$

Secondly, for  $TC$ . We know that there is a fixed tax expense of \$200 ( $FC = 200$ ) and a variable cost of \$980 per employee for wages ( $980x$ ). So

$$\begin{aligned} TC &= VC + FC \\ &= 980x + 200 \end{aligned} \quad (7.5)$$

Now that we have  $TR$  and  $TC$  we can find the break-even point by letting  $TR = TC$ .

$$\begin{aligned} TR &= TC \\ 1000x &= 980x + 200 && \text{Subtract 980 from both sides} \\ 20x &= 200 \\ \frac{20x}{20} &= \frac{200}{20} && \text{Divide both sides by 20} \\ x &= 10 && \text{This is our } x \text{ value at the break-even point} \end{aligned}$$

Now recall that the break-even point involves neither loss nor profit. Note that the question in the problem was what is the minimum number of employees required to be profitable. Thus we would need at least 11 employees to be profitable.

To express the break-even point as a percent of capacity we divide

$$\frac{x \text{ value at break-even point}}{x \text{ value at capacity}} \quad (7.6)$$

or

$$\frac{y \text{ value at break-even point}}{y \text{ value at capacity}} \quad (7.7)$$

depending on what is given in the problem. We use the first expression here.

$$\frac{x \text{ value at break-even point}}{x \text{ value at capacity}} = \frac{10}{20} = 0.5 = 50\%$$

Concluding our problem. ◀

**Example 7.1.3.** Draw a break-even chart for the given cost and revenue functions. Assume that  $x = 25$  is capacity.

$$\begin{cases} TC = x + 15 \\ TR = 3x \end{cases}$$

**Solution:** We begin with the fact that the  $y$ -intercepts are  $(0, 15)$  for  $TC$  and  $(0, 0)$  for  $TR$ . The break-even point is found by setting  $TR$  and  $TC$  equal:

$$\begin{aligned} TR &= TC \\ 3x &= x + 15 && \text{Subtract } x \text{ from both sides} \\ 2x &= 15 \\ \frac{2x}{2} &= \frac{15}{2} && \text{Divide both sides by 2} \\ x &= 7.5 \end{aligned}$$

To find the  $y$ -value, we substitute this into  $TR$ .

$$\begin{aligned} TR &= 3x \\ &= 3(7.5) \\ &= 22.5 \end{aligned}$$

Thus the break-even point is at  $(7.5, 22.5)$ .

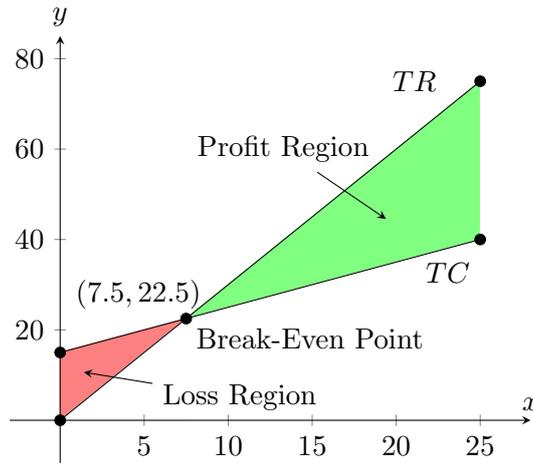
We also need the  $y$ -values at capacity. We can find this by substituting  $x = 25$  into both  $TR$  and  $TC$ .

$$\begin{aligned} TC &= x + 15 = 25 + 15 = 40 \\ TR &= 3x = 3(25) = 75 \end{aligned}$$

We have more points:  $(25, 40)$  on  $TC$  and  $(25, 75)$  on  $TR$ .

Finally, we have a summary of all the points and the chart.

	Points to be Graphed
TR	$(0,0)$ $(7.5,22.5)$ $(25,75)$
TC	$(0,15)$ $(7.5,22.5)$ $(25,40)$



and that finishes the problem. ◀

### 7.1.2 Other Break-Even Analysis Problems

The next three problems are generally optional within BMAT 220 and should be considered more challenging.

**Example 7.1.4.** The following information is available about a corporation for the current year. Sales \$40000, Fixed costs, \$12600, Variable costs \$16000 and Capacity at sales maximum of \$60000. Find the total revenue and total cost functions. Find the break-even point in sales dollars and as a percent of capacity.

**Solution:** Our first objective is to find the Total Revenue ( $TR$ ) and Total Cost ( $TC$ ) functions. In constructing  $TR$  and  $TC$  we will let  $x$  be the total dollar value of the product sold. We do this because we do not know the price per unit or cost per unit from the problem. Thinking about this carefully gives us:

$$TR = x$$

Now to find  $TC$ . We know that  $TC = VC + FC$ , where  $VC$  is Variable Costs and  $FC$  is Fixed Costs. We know from the problem that  $FC = 12600$ . To find the variable cost we need to consider how many cents of every dollar of revenue goes to variable costs. To find this compute:

$$\frac{\text{Total Variable Cost}}{\text{Total Revenue}} = \frac{16000}{40000} = 0.4 = 40\%$$

Computed from the data in the problem. This tells us that for every dollar of money incoming, 40 cents must go to variable costs. Further, since  $x$  is the dollar value of the units sold:

$$VC = 0.4x$$

Giving us:

$$\begin{aligned}TC &= VC + FC \\ &= 0.4x + 12600\end{aligned}$$

So we have our Total Revenue and Total Cost Functions (parts 1a and 1b in the problem). On to part 2.

From this point on the problem is exactly like all of the other Break-Even problems we have done. We now need to do the break-even analysis. Part (a) wants us to find the break-even point in terms of sales dollars. To do this set  $TR = TC$  and solve.

$$\begin{aligned}TR &= TC \\ x &= 0.4x + 12600 \\ 1x &= 0.4x + 12600 \\ 1x - 0.4x &= 12600 \\ 0.6x &= 12600 \\ x &= \frac{12600}{0.6} \\ x &= 21000\end{aligned}$$

This tells us that the break-even point in terms of dollars is \$21 000.

Next in part (2b), we need to find the break-even point as a percent of capacity. This is the easy part. To find this simply compute:

$$\begin{aligned}\frac{\text{Break-Even Point in Terms of Dollars}}{\text{Sales Value at Capacity}} &= \frac{21000}{60000} \\ &= 0.35 \\ &= 35\%\end{aligned}$$

Notice that from the original problem it says that the capacity is at sales maximum of \$60 000. This is how we know "Sales Value at Capacity" in the formula above. ◀

**Example 7.1.5.** The following data pertain to the operating budget of Matt Manufacturing. Capacity per period is sales of \$800,000. In a period Sales were \$720,000, Fixed Costs were \$220,000, Total Variable Cost was \$324,000. Find the Total Cost Function, Total Revenue Function, Break-even point in terms of dollars, and Break-even point as a percent of Capacity.

**Solution:** This will all be very similar to the previous problem. Our first objective is to find the Total Revenue ( $TR$ ) and Total Cost ( $TC$ ) functions. In constructing  $TR$  and  $TC$  we will let  $x$  be the total dollar value of the product sold. We do this because we do not know the price per unit or cost per unit from the problem. Thinking about this carefully gives us:

$$TR = x$$

Now to find  $TC$ . We know that  $TC = VC + FC$ , where  $VC$  is Variable Costs and  $FC$  is Fixed Costs. We know from the problem that  $FC = 220000$ . To find the variable cost we need to consider how many cents of every dollar of revenue goes to variable costs. To find this compute:

$$\frac{\text{Total Variable Cost}}{\text{Total Revenue}} = \frac{324000}{720000} = 0.45 = 45\%$$

Computed from the data in the problem. This tells us that for every dollar of money incoming, 45 cents must go to variable costs. Further, since  $x$  is the dollar value of the units sold:

$$VC = 0.45x$$

Giving us:

$$\begin{aligned} TC &= VC + FC \\ &= 0.45x + 220000 \end{aligned}$$

So we have our Total Revenue and Total Cost Functions. From this point on the problem is exactly like all of the other Break-Even problems we have done. We now need to do the break-even analysis. We will now find the break-even point in terms of sales dollars. To do this set  $TR = TC$  and solve.

$$\begin{aligned} TR &= TC \\ x &= 0.45x + 220000 \\ 1x &= 0.45x + 220000 \\ 1x - 0.45x &= 220000 \\ 0.55x &= 220000 \\ x &= \frac{220000}{0.55} \\ x &= 400000 \end{aligned}$$

This tells us that the break-even point in terms of dollars is \$400 000. Next, to find the break-even point as a percent of capacity. This is the easy part. To find this simply compute:

$$\begin{aligned} \frac{\text{Break-Even Point in Terms of Dollars}}{\text{Sales Value at Capacity}} &= \frac{400000}{800000} \\ &= 0.50 \\ &= 50\% \end{aligned}$$

Notice that from the original problem it says that the capacity is at sales maximum of \$800,000. ◀

**Example 7.1.6.** Harrow Seed and Fertilizer compiled the following estimates for its operations. Capacity per period is sales of \$150,000. Sales during a specific period

were \$120,000, Fixed Costs were \$43,200, Total Variable Cost was \$48,000. Find the Total Cost Function, Total Revenue Function, Break-even point in terms of dollars, and Break-even point as a percent of Capacity.

**Solution:** Our first objective is to find the Total Revenue ( $TR$ ) and Total Cost ( $TC$ ) functions. In constructing  $TR$  and  $TC$  we will let  $x$  be the total dollar value of the product sold. We do this because we do not know the price per unit or cost per unit from the problem. Thinking about this carefully gives us:

$$TR = x$$

Now to find  $TC$ . We know that  $TC = VC + FC$ , where  $VC$  is Variable Costs and  $FC$  is Fixed Costs. We know from the problem that  $FC = 43200$ . To find the variable cost we need to consider how many cents of every dollar of revenue goes to variable costs. To find this compute:

$$\frac{\text{Total Variable Cost}}{\text{Total Revenue}} = \frac{48000}{120000} = 0.4 = 40\%$$

Computed from the data in the problem. This tells us that for every dollar of money incoming, 40 cents must go to variable costs. Further, since  $x$  is the dollar value of the units sold:

$$VC = 0.4x$$

Giving us:

$$\begin{aligned} TC &= VC + FC \\ &= 0.4x + 43200 \end{aligned}$$

So we have our Total Revenue and Total Cost Functions. From this point on the problem is exactly like all of the other Break-Even problems we have done. We now need to do the break-even analysis. We will now find the break-even point in terms of sales dollars. To do this set  $TR = TC$  and solve.

$$\begin{aligned} TR &= TC \\ x &= 0.4x + 43200 \\ 1x &= 0.4x + 43200 \\ 1x - 0.4x &= 43200 \\ 0.6x &= 43200 \\ x &= \frac{43200}{0.6} \\ x &= 72000 \end{aligned}$$

This tells us that the break-even point in terms of dollars is \$72 000. Next, to find the break-even point as a percent of capacity. This is the easy part. To find this simply

compute:

$$\begin{aligned}\frac{\text{Break-Even Point in Terms of Dollars}}{\text{Sales Value at Capacity}} &= \frac{72000}{150000} \\ &= 0.48 \\ &= 48\%\end{aligned}$$

Notice that from the original problem it says that the capacity is at sales maximum of \$150 000. ◀

## 7.2 Contribution Margin and Contribution Rate

This section corresponds to section 6.2 in the course textbook. Our interest here is a more detailed look at the mathematics behind Cost Volume Profit Analysis and how we may take advantage of it to solve new problems and look at old problems in a different way.

To begin let us consider looking at some mathematics at the break-even point. We begin by recalling that  $TR = px$  and  $TC = cx + f$ , where  $c$  is the total variable cost per unit,  $p$  is the sell price per unit,  $f$  is the total fixed cost, and  $x$  is the volume. With this we can define profit as

$$\begin{aligned}PFT &= TR - TC \\&= px - (cx + f) \\&= px - cx - f \\&= (p - c)x - f\end{aligned}\tag{7.8}$$

If we assume we are at break-even volume, this would imply that our profit is 0. So proceeding from the work above we have

$$\begin{aligned}0 &= (p - c)x - f \\f &= (p - c)x\end{aligned}\tag{7.9}$$

$$\frac{f}{p - c} = x\tag{7.10}$$

The most important thing to notice about the work above is since  $x$  is the break-even volume, this says we can compute it simply by computing the left hand side of Equation 7.10, that is, fixed cost divided by the difference of the variable cost per unit and the sell price per unit. We no longer have to solve a system of equations as with the previous section.

There are other labels the financial community will use with respect to some of the expressions used above. First, the **Contribution Margin per unit**,  $M_u$ , is the amount in parenthesis in Equation 7.9. Thus

$$M_u = p - c\tag{7.11}$$

If we now multiply both sides of Equation 7.11 by  $x$  we get what is called the **Total Contribution Margin** or  $M_t$ . So

$$M_t = M_u x = (p - c)x$$

which should be recognized as the right hand side of equation 7.9.

Further, in finance theory we define another parameter called the **Contribution Rate** or  $M_r$  and define it as

$$M_r = \frac{M_u}{p} = \frac{p - c}{p}\tag{7.12}$$

One last concern is how might we compute the break-even sales (in sales dollars)? Well, we know that at the break-even point the total revenue is  $TR = px$  and note that if we multiply both sides of Equation 7.10 above by  $p$  this will give us another expression for computing the break-even sales. So

$$\begin{array}{ll}
 x = \frac{f}{p - c} & \text{Equation 7.10} \\
 px = \frac{pf}{p - c} & \text{Multiply both sides by } p \\
 px = \frac{p}{p - c} f & \text{Rearrange Numerator} \\
 px = \frac{f}{\frac{p-c}{p}} & \text{Rewrite as a Complex Fraction}
 \end{array}$$

Note here that the global denominator in the last line is the contribution rate,  $M_r$  and remember that  $px$  is the total revenue at the break-even point (making it the break-even sales) we conclude that

$$\text{Break-Even Sales} = \frac{f}{M_r} \quad (7.13)$$

Many of the above relations can be useful to solve problems in break-even analysis, especially Equations 7.8, 7.10, and 7.13.

**Example 7.2.1.** [1, p. 245] Market research for a new product called the Scrubby indicates that the product can be sold at \$50 per unit. Cost analysis provides the following information. Fixed cost per period = \$8640, Variable cost per unit = \$30, and Production capacity per period = 900 units. Use contribution margin per unit to determine the contribution rate and the break-even volume and break-even sales.

**Solution:** Given the information above we just need to substitute into the equations above and compute. First, note that  $f = 8640$ ,  $p = 50$ , and  $c = 30$ . So the Contribution margin per unit will be

$$\begin{aligned}
 M_u &= p - c \\
 &= 50 - 30 \\
 &= \$20
 \end{aligned}$$

Secondly, the Contribution rate is

$$\begin{aligned}
 M_r &= \frac{M_u}{p} \\
 &= \frac{20}{50} \\
 &= 0.4 \\
 &= 40\%
 \end{aligned}$$

Next, the break even volume is

$$\begin{aligned}x &= \frac{f}{p - c} \\ &= \frac{8640}{50 - 30} \\ &= 432\end{aligned}$$

So break-even volume is 432 units.

Finally, we use Equation 7.13 to find the break-even sales. So

$$\begin{aligned}\text{Break-Even Sales} &= \frac{f}{M_r} \\ &= \frac{8640}{0.4} \\ &= 21600\end{aligned}$$

So the break-even sales are \$21,600, concluding our problem. ◀

**Example 7.2.2.** [1, p. 242] Tara makes and sells hats for children and adults. She is able to sell the hats for \$18 per unit. Materials for the hats cost \$4 each. She has fixed costs per month of \$280, and estimates that she can make and sell 80 hats each month. How many hats does Tara need to sell to break-even? What is the break-even point in terms of sales dollars?

**Solution:** To work this problem we could certainly use the materials from the previous section, however, let us try working the problem using what we have learned here.

To find the break-even volume for the first question we can use Equation 7.10 with  $f = 280$ ,  $p = 18$ , and  $c = 4$ . So

$$\begin{aligned}x &= \frac{f}{p - c} \\ &= \frac{280}{18 - 4} \\ &= 20 \text{ hats}\end{aligned}$$

The second question wants to know the break-even point in terms of sales dollars. We can use Equation 7.13 to find this, but first we need to know the contribution rate,  $M_r$ . We can use Equation 7.12 for that.

$$\begin{aligned}M_r &= \frac{p - c}{p} && \text{Equation 7.12} \\ &= \frac{18 - 4}{18} \\ &= 0.\bar{7}\end{aligned}$$

We can now substitute this into Equation 7.13 to give

$$\begin{aligned}\text{Break-Even Sales} &= \frac{f}{M_r} && \text{Equation 7.13} \\ &= \frac{280}{0.\bar{7}} \\ &= 360\end{aligned}$$

So the break-even sales are \$360 concluding our problem. ◀

In bringing this section to a close students are reminded that methods from either of 7.1 or 7.2 can be used to solve these problems. In 7.2 we are trying to give some alternatives with the goal of potentially shortening some of the work.

### 7.3 Introduction to Sensitivity Analysis

This section corresponds to section 6.3 in the textbook and is very much related to the topics in the previous two sections. In particular we are interested how changing a parameter in the cost and/or revenue functions effects the break-even values and profit. This process is called **What-if analysis** or **Sensitivity Analysis**.

**Example 7.3.1.** [1, p. 249] For Eric's birdhouse business, recall that the unit price is \$30 and the fixed cost is \$400. If the material costs increase to \$12 for each birdhouse, what is the new break-even point in terms of units? How much does this change the break-even point in terms of units from the example in section 7.1? (See example 6.1A on page 232 of textbook)

**Solution:** To work this problem we can again use Equation 7.10 with  $f = 400$ ,  $p = 30$ , and  $c = 12$ . So

$$\begin{aligned}x &= \frac{f}{p - c} \\ &= \frac{400}{30 - 12} \\ &= 22.\bar{2}\end{aligned}$$

So the break-even number of birdhouses is about 22.

In answering the question about how much this changed from the previous material costs we need to look back at example 6.1A in the textbook on page 234. We see that the break-even volume here was 20 birdhouses. Thus the change in material costs increased the break-even volume by  $22 - 20 = 2$  units. ◀

**Example 7.3.2.** [1, p. 249] For Eric's birdhouse business, if the material cost remains at the original \$10, the rent cost is decreased to \$300, and he sold 50 birdhouses, what would be the resulting profit?

**Solution:** To solve this problem we can use Equation 7.8 in the previous section with  $f = 300$ ,  $p = 30$ ,  $x = 50$ , and  $c = 10$ . So

$$\begin{aligned}PFT &= (p - c)x - f && \text{Equation 7.8} \\ &= (30 - 10)50 - 300 \\ &= 1000\end{aligned}$$

So the profit would be \$1000. ◀

As you are working these problems take careful note as to the exact wording involved. For example, the problem may say "the fixed cost is decreased to \$800", which means  $f = 800$ , however, other language may include something like "the fixed cost is decreased by \$800," which means to take the original fixed cost and subtract 800 from it. Be very carefully with this as it results in a significant source of error for students on tests and quizzes.

## 8 Discount Series and Markup

In this chapter we take a look at the topics of discount, markup, and markdown. Many of these topics have been discussed at length in BMAT 110 for those that took it. It might be a good idea to review sections that involve percentage decrease and increase, which can be found in chapter 3 of the course textbook. This chapter corresponds to chapter 5 in the course textbook.

### 8.1 Discount Series

This section corresponds to section 5.1 in the textbook.

#### 8.1.1 Net price factor method

You may recall working several discount problems earlier in the course. Among these problems you may have encountered several formulae

$$\text{Amount of Discount} = \text{List Price} \times \text{Rate of Discount}$$

or

$$A = L \times d \tag{8.1}$$

From this equation we can derive other formulae:

$$L = \frac{A}{d} \tag{8.2}$$

$$d = \frac{A}{L} \tag{8.3}$$

You may recall that the above formulae are used to compute the amount of discount and that if we wish to compute to new price we have to subtract the amount of discount from the list price. In mathematics

$$N = L - Ld \quad \text{where } N \text{ is the new price} \tag{8.4}$$

This is a completely useful form for computing a discount provided there is only one discount involved. Thus a natural question is: what happens if there is more than one discount on a list price? We call such a situation a **Discount Series**, and is the primary topic of this section.

To answer the question above with formula 8.4 we would have to do a lot of complicated arithmetic. In fact, the arithmetic would become significantly more complicated as we

add more discounts. However, there is a nice alternative. We call this alternative the **Factor Form** of equation 5.4 and now derive it.

$$\begin{aligned}
 N &= L - Ld \\
 &= L \underbrace{(1 - d)}_{\text{net price factor of } d \text{ (NPF)}} && \text{Factor out } L && (8.5)
 \end{aligned}$$

Equation 8.5 is the net factor equivalent of equation 8.4.

This might not seem very significant and a bit anti-climactic, but let us consider an example to motivate this a bit further.

**Example 8.1.1.** Using equation 8.5, find the price paid at the cashier on an item with a list price of \$69.99 with a discount of 10%.

**Solution:** Here  $N$  is the value we are looking for, while  $d = 0.10$ , and  $L = 69.99$ . So

$$\begin{aligned}
 N &= L(1 - d) \\
 &= 69.99(1 - 0.10) \\
 &= 69.99(0.90) \\
 &= 62.991 \\
 &= \$62.99
 \end{aligned}$$

So our answer is \$62.99. ◀

This should seem easier than using equation 8.4. It turns out it is far easier when dealing with more than one discount.

**Property 1.** Let some list price  $L$  be subject to the discount series  $d_1, d_2, d_3 \dots d_n$ , then the net price  $N$  is

$$N = L \underbrace{(1 - d_1)}_{\text{NPF of discount 1}} \underbrace{(1 - d_2)}_{\text{NPF of discount 2}} \underbrace{(1 - d_3)}_{\text{NPF of discount 3}} \dots \underbrace{(1 - d_n)}_{\text{NPF of discount } n} \quad (8.6)$$

**Example 8.1.2.** Find the net price of a list price total of \$1203, subject to the discount series 7%, 5%, and 3%. Round your answer to the nearest cent.

**Solution:** Here we proceed with property 3.

$$\begin{aligned}
 N &= L(1 - d_1)(1 - d_2)(1 - d_3) \\
 &= 1203(1 - 0.07)(1 - 0.05)(1 - 0.03) \\
 &= 1203(0.93)(0.95)(0.97) \\
 &= 1030.964985 \\
 &= \$1030.96
 \end{aligned}$$

So our net price is \$1030.96. ◀

**Example 8.1.3.** Eli's company makes windows and purchases materials from a local company, owned by Herb, who Eli has known for many years. As a result of the close relationship Eli receives discounts on the material he purchases. First, Eli always purchases in bulk which gives him a discount of 9%. Secondly, his workshop is nearby which effectively amounts to a discount for shipping of 6%. Lastly, since Eli and the materials company owner are good friends he gets a 2% discount. If Eli purchases materials totaling \$4,832, how much will he actually pay?

**Solution:** Here, again, we proceed with property 1.

$$\begin{aligned}
 N &= L(1 - d_1)(1 - d_2)(1 - d_3) \\
 &= 4832(1 - 0.09)(1 - 0.06)(1 - 0.02) \\
 &= 4832(0.91)(0.94)(0.98) \\
 &= 4050.626944 \\
 &= \$4050.63
 \end{aligned}$$

So he pays \$4050.63. ◀

### 8.1.2 Single equivalent rate of discount

It is sometimes useful to understand that any discount series can be interpreted as a single discount or single equivalent rate of discount. We find this with the following definition.

**Definition 2.** Let  $d_1, d_2, d_3 \dots d_n$  be some discount series of  $n$  discounts, then the **Single Equivalent Rate of Discount** is

$$\begin{aligned}
 \text{Single Equivalent Rate of Discount} &= 1 - (1 - d_1)(1 - d_2)(1 - d_3) \dots (1 - d_n) \\
 &= 1 - \text{Product of the NPF's}
 \end{aligned}$$

**Example 8.1.4.** Find the single equivalent rate of discount in example 8.1.2. The discount series was 7%, 5%, and 3%. Round to the nearest percent.

**Solution:** From definition 12 above

$$\begin{aligned}
 \text{Single Equivalent Rate of Discount} &= 1 - (1 - d_1)(1 - d_2)(1 - d_3) \\
 &= 1 - (1 - 0.07)(1 - 0.05)(1 - 0.03) \\
 &= 1 - (0.93)(0.95)(0.97) \\
 &= 0.143005 \\
 &= 14\%
 \end{aligned}$$

So the single equivalent rate of discount is 14%. ◀

**Example 8.1.5.** Find the single equivalent rate of discount in example 8.1.3. The discount series was 9%, 6%, and 2%. Round to the nearest percent.

**Solution:** From definition 12

$$\begin{aligned}
 \text{Single Equivalent Rate of Discount} &= 1 - (1 - d_1)(1 - d_2)(1 - d_3) \\
 &= 1 - (1 - 0.09)(1 - 0.06)(1 - 0.02) \\
 &= 1 - (0.91)(0.94)(0.98) \\
 &= 0.161708 \\
 &= 16\%
 \end{aligned}$$

So the single equivalent rate of discount is 16%. ◀

This final example is meant to be somewhat of a challenge, but could show up on an exam or quiz.

**Example 8.1.6.** Referencing example 5.1.3 with Eli's company, consider a situation where a competitor company comes in and offers Eli a single discount of 17% on purchases, if he will buy from them. If Herb wants to keep Eli's business he must match the new company's discount. What fourth discount should Herb provide? Assume Eli is purchasing \$5000 worth of materials and round the answer to the nearest percent.

**Solution:** Here we will use property 3 and note that we want the net price from the two business to be the same and note that we need a variable to find the fourth discount from Herb's business. We need to find this  $d_4$ .

$$\begin{aligned}
 L(1 - d_1)(1 - d_2)(1 - d_3)(1 - d_4) &= L(1 - d_{\text{competitor}}) \\
 5000(1 - 0.09)(1 - 0.06)(1 - 0.02)(1 - d_4) &= 5000(1 - 0.17) \\
 5000(0.91)(0.94)(0.98)(1 - d_4) &= 5000(0.83) && \text{Compute NPF's} \\
 4191.46(1 - d_4) &= 4150 && \text{Multiply} \\
 \frac{4191.46(1 - d_4)}{4191.46} &= \frac{4150}{4191.46} && \text{Divide both sides by 4191.46} \\
 1 - d_4 &= 0.990108458 \\
 1 - 0.990108458 &= d_4 && \text{Isolate } d_4 \\
 0.00989154\dots &= d_4 \\
 1\% &= d_4 && \text{Round after multiplying by 100}
 \end{aligned}$$

So Herb would have to include an additional discount of 1%. ◀

## 8.2 Payment Terms and Cash Discounts

This section corresponds to section 5.2 in the textbook. There is very little in this section that will be new to students with the exception of payment term notation involved in manufacturing, wholesales, distributing, and retail. We will outline how to read the notation and then go over a few examples. We will be drawing on the mathematics of simple interest and basic percent which should be easily understood by students especially if the BMAT 110 course had been recently taken.

Let us begin by looking at **standard payment term** notation. Note that the following notation is strictly used by the fields listed above and do not have a deep mathematical meaning.

**Definition 3.** *Notation of the form;  $d/p_d, n/p_c$ ; is called **ordinary dating** and is used to indicate invoice payment terms. In the notation,  $d$  is the cash discount percent given if payment is received before the lapse of  $p_d$ ,  $p_d$  is the length of the discount period,  $n$  is the net amount owed on the invoice, and  $p_c$  is the length of the credit period.*

[1, p. 187]The most commonly used payment terms in business are:  $2/10, n/30$  (read "two ten, net thirty"). This means that if the buyer pays the debt off within 10 days, they will receive a 2% discount, otherwise they must pay the total amount,  $n$ , within 30 days. We will work with several forms of discount terms for the sake of practice.

**Example 8.2.1.** Sandeep receives payment terms on his invoice from Home Hardware, where he owes \$1200. The payment terms are dated August 2nd and the terms are  $2/10, n/30$ , if the invoice is paid on August 23rd how much will be paid? How much will be paid if payment is received on August 8th?

**Solution:** The payment terms mean that if Sandeep makes the payment within 10 days, then he will receive a 2% discount on the amount owed. If he cannot make the payment within 10 days, but within 30 days, then he must pay the full \$1200.

To answer the first question, note that the payment will be made on August 23rd, 21 days after the invoice was dated, thus to pay the balance of the invoice he must pay the full amount. So \$1200 is our answer.

To answer the second question, note that the payment will be made on August 8th, 6 days after the invoice was dated thus Sandeep will receive a 2% discount. Thus our answer here is  $1200 \times 0.98 = \$1176$  (using the NPF method). ◀

**Example 8.2.2.** What amount must be paid if invoices dated January 6th for \$430, January 8th for \$689, and January 15th for \$598 all with the terms  $4/15, n/35$ ? The entire debt will be paid on January 22nd.

**Solution:** The payment terms here mean that if the debt for an invoice is paid within 15 days, then that particular debt will receive a 4% discount, otherwise the entire debt must be paid within 35 days. So let us look at each invoice one at a time. The January 6th invoice will be paid outside of the discount period because  $6 + 15 = 21$ st day of the month, so no discount on the \$430. However, the other two invoices will be paid within the discount period because  $8 + 15 = 23$ rd day of the month and  $15 + 15 = 30$ th day of the month, thus we need to apply a 4% discount to the last two invoices and add everything up. So  $430 + 0.96(689) + 0.96(598) = \$1665.52$  is the amount that should be paid, if paid off on January 22nd. ◀

**Example 8.2.3.** What amount must be paid if invoices dated April 1st for \$1000, April 4th for \$951, and April 5th for \$850 all with the terms  $3/12, n/40$ ? The entire debt will be paid on April 17th.

**Solution:** The payment terms here mean that if the debt for an invoice is paid within 12 days, then that particular debt will receive a 3% discount, otherwise the entire debt must be paid within 40 days. So let us look at each invoice one at a time. The April 1st invoice will be paid outside of the discount period because  $1 + 12 = 13$ th day of the month, so no discount on the \$1000. The next invoice will also be paid outside of the discount period because  $4 + 12 = 16$ th day of the month. However, the last invoice will be paid within the discount period because  $5 + 12 = 17$ th day of the month, thus we need to apply a 4% discount to the last invoice and add everything up. So  $1000 + 951 + 0.97(850) = \$2775.50$  is the amount that should be paid, if paid off on April 17th. ◀

## 8.3 Markup

This section corresponds to section 5.3 in the textbook. Here we are interested in how to precisely markup products for sale that have been purchased from a third party after all expenses and costs are taken into account. Alternatively, we might be purchasing many products to assemble then sale.

### 8.3.1 Some basic concepts

**Definition 4.** Let  $c$  be the cost to buy some product,  $e$  be the expenses or overhead of the business, and  $p$  be the profit required by the business owner; then the **selling price** of the product is

$$s = c + e + p \quad (8.7)$$

We can manipulate this equation algebraically to get some other useful formulae.

$$s - c = e + p \quad (8.8)$$

We sometimes call the expense added to profit the markup,  $m$ .

$$m = e + p \quad (8.9)$$

$$s = c + m \quad (8.10)$$

**Example 8.3.1.** Vlad owns a wood shop and wants to sell cut lumber for \$600 per ton to compete with other businesses in the area. After some research he has determined that he can purchase the lumber from a mill at a cost of \$350 per ton and his overhead will amount to about 15% of the cost. Under these circumstances, how much profit can he make?

**Solution:** Here we use equation 8.7 with  $s = 600$ ,  $c = 350$ ,  $e = 0.15c$ , we are looking

for  $p$ . Putting all of this together we have.

$$\begin{aligned} s &= c + e + p \\ 600 &= 350 + 0.15(350) + p && \text{Substitute in } c \text{ and } e \\ 600 &= 402.5 + p \\ \$197.50 &= p && \text{Subtract } 402.5 \text{ from both sides} \end{aligned}$$

So our answer is \$197.50. ◀

**Example 8.3.2.** Nihkita has two options to fill a spot on a shelf in her store. Item A costs \$59 and sells for \$99. Item B costs \$52 and sells for \$102. Overhead of the business is 20% of the cost. Determine the markup, expense, and profit for each item. Which product would be better to have on the shelf in terms of profit?

**Solution:** We start with equation 8.10 to find the markup and use subscripts to denote the item for each calculation.

$$\begin{aligned} s_a &= c_a + m_a \\ 99 &= 59 + m_a && \text{Substitute in } c \text{ and } s \\ \$40 &= m_a \end{aligned}$$

$$\begin{aligned} s_b &= c_b + m_b \\ 102 &= 52 + m_b && \text{Substitute in } c \text{ and } s \\ \$50 &= m_b \end{aligned}$$

Now for the expense. We know that the overhead (expense) is 20% of the cost. or  $e = 0.20c$ .

$$\begin{aligned} e_a &= 0.20c_a \\ &= 0.20(59) \\ &= \$11.80 \end{aligned}$$

$$\begin{aligned} e_b &= 0.20c_b \\ &= 0.20(52) \\ &= \$10.40 \end{aligned}$$

Finally, from equation 8.9 we know that  $p = m - e$ .

$$\begin{aligned} p_a &= m_a - e_a \\ &= 40 - 11.80 \\ &= \$28.20 \end{aligned}$$

$$\begin{aligned} p_b &= m_b - e_b \\ &= 50 - 10.40 \\ &= \$39.60 \end{aligned}$$

Item B would be better to have on the shelf because it will generate more profit. ◀

### 8.3.2 Rate of markup

Markup is usually interpreted in two different ways:

1. As a percent of cost.
2. As a percent of selling price.

So we have two different definitions:

**Definition 5.** Let  $m$  be the markup,  $c$  be the cost,  $s$  be the selling price,  $r_c$  be the **rate of markup based on cost**, and  $r_s$  be the **rate of markup based on selling price**, then

$$r_s = \frac{m}{s} \quad (8.11)$$

and

$$r_c = \frac{m}{c} \quad (8.12)$$

**Example 8.3.3.** Compute (a) the missing value (cost, selling price, or markup); (b) the rate of markup based on cost; and (c) the rate of markup based on selling price for each of the following:

- (i) Cost is \$50; selling price is \$75
- (ii) Selling price is \$95; markup is \$15
- (iii) Cost is \$60; markup is \$8

**Solution:** In each problem we first need to find the missing value using equation 8.10, then substitute into the appropriate formula from definition 14 above.

- (i) In this problem the missing value is the markup. So

$$\begin{aligned} s &= c + m \\ 75 &= 50 + m \\ 25 &= m \end{aligned} \quad \text{Subtract 50 from both sides}$$

Now to find  $r_s$  and  $r_c$ .

$$\begin{aligned} r_s &= \frac{m}{s} \\ &= \frac{25}{75} \\ &= 0.\overline{33} \\ &\approx 33\% \end{aligned}$$

and

$$\begin{aligned}r_c &= \frac{m}{c} \\ &= \frac{25}{50} \\ &= 0.5 \\ &= 50\%\end{aligned}$$

(ii) Here the missing value is cost. So

$$\begin{aligned}s &= c + m \\ 95 &= c + 15 \\ 80 &= c\end{aligned}$$

Subtract 15 from both sides

Now to find  $r_s$  and  $r_c$ .

$$\begin{aligned}r_s &= \frac{m}{s} \\ &= \frac{15}{95} \\ &= 0.1578947\dots \\ &\approx 16\%\end{aligned}$$

and

$$\begin{aligned}r_c &= \frac{m}{c} \\ &= \frac{15}{80} \\ &= 0.1875 \\ &\approx 19\%\end{aligned}$$

(iii) Here the missing value is selling price. So

$$\begin{aligned}s &= c + m \\ &= 60 + 8 \\ &= 68\end{aligned}$$

Now to find  $r_s$  and  $r_c$ .

$$\begin{aligned}r_s &= \frac{m}{s} \\ &= \frac{8}{68} \\ &= 0.117647\dots \\ &\approx 12\%\end{aligned}$$

and

$$\begin{aligned}r_c &= \frac{m}{c} \\ &= \frac{8}{60} \\ &= 0.1\bar{3} \\ &\approx 13\%\end{aligned}$$

This concludes our problem. ◀

## 8.4 Markdown

This section corresponds to section 5.4 in the course textbook. Our interest here is the opposite of our interest in the previous section. How should we effectively markdown prices for goods or services in a business environment. Here we will use many of the topics discussed previously in this chapter including discount and the formulae involved in sell price.

The most fundamental concept in this section is the notation of **Total Cost** which is different from *Cost* in the previous section. We define total cost,  $T$ , by the equation

$$T = \text{Cost of Buying} + \text{Expenses} \quad (8.13)$$

and note that with respect to Equation 8.7 in section 8.3 we have

$$S = T + P$$

with  $s$  being the sell price and  $p$  the profit.

With this in mind, we create two notions of *Loss*. The first we will call **operating loss** - when the price is not large enough to cover the total cost, i.e.  $S < T$ . The other type of loss we will call **absolute loss** - when the sell price is not enough to cover the cost of buying,  $c$  in Equation 8.3, i.e.  $S < C$

As a result of the above construction, businesses may choose to markdown the regular selling price of some item for several reasons. So if we let  $S$  be the regular selling price,  $MD$  be the markdown, and  $N$  be the new selling price we have

$$N = S - MD \quad (8.14)$$

Further, we will refer to the quotient formed by dividing the markdown by the regular selling price as the markdown rate. So

$$\text{Markdown Rate} = \frac{MD}{S} \quad (8.15)$$

It is not difficult to see that

$$N = S \times \text{NPF of the Markdown Rate} \quad (8.16)$$

The markdown rate will often times be given within problems but not always.

**Example 8.4.1.** [1, p. 204] The Cook Nook paid \$115.24 for a set of dishes. Expenses are 18% of selling price and the required profit is 15% of selling price. During an inventory sale, the set of dishes was marked down 30%

1. What was the regular selling price?
2. What was the sale price?
3. What is the operating profit or loss?

**Solution:** Using Equation 8.7 from section 8.3 with  $E = 0.18S$  and  $P = 0.15S$  we have

$$\begin{aligned} S &= C + E + P \\ S &= 115.24 + 0.18S + 0.15S \\ S &= 115.24 + 0.33S \\ 0.67S &= 115.24 \\ S &= \frac{115.24}{0.67} \\ S &= 172 \end{aligned}$$

Thus the regular sell price was \$172.00. This answers question 1.

Now to find the sale price. We can find this now that we know the regular sell price and the markdown rate using Equation 8.14 or 8.16. Here we choose to use Equation 8.16. So

$$\begin{aligned} N &= S \times \text{NPF of the Markdown Rate} \\ N &= 172 \times 0.70 \\ N &= 120.40 \end{aligned}$$

So the new sale price is \$120.40

Finally, for question 3 using what information we have accumulated above we need to find the operating profit or loss. Note that from our discussion above on operating loss, we need to know the total cost. We can find this using Equation 8.13. Thus

$$\begin{aligned} T &= C + E \\ &= C + 0.18S \\ &= 115.24 + 0.18(172) \\ &= 146.20 \end{aligned}$$

So our total cost is \$146.20.

Note now that our total cost,  $T$ , is larger than our sale price,  $N$ . So we have an operating loss of  $146.20 - 120.40 = \$25.80$ . This concludes our problem. ◀

**Example 8.4.2.** [1, p. 205] Lund Sporting Goods sold a bicycle regularly priced at \$195 for \$144.30.

1. What is the amount of markdown?
2. What is the rate of markdown?

**Solution:** We begin by finding the amount of markdown. We can find this using Equation 8.14. So

$$\begin{aligned} N &= S - MD \\ 144.30 &= 195 - MD \\ -50.7 &= -MD \\ 50.7 &= MD \end{aligned}$$

So the markdown is \$50.70.

Next, we want to find the rate of markdown. We can use our work above in conjunction with Equation 8.15 to achieve a solution. Thus

$$\begin{aligned} \text{Markdown Rate} &= \frac{MD}{S} \\ &= \frac{50.7}{195} \\ &= 0.26 \end{aligned}$$

Thus the markdown rate is 26%. ◀

**Example 8.4.3.** [1, p. 206] The Winemaker sells Okanagan concentrate for \$22.50. The store's overhead expenses are 50% of cost and the owners required profit is 30% of cost.

1. For how much does The Winemaker buy the concentrate?
2. What is the price needed to cover all of the costs and expenses?
3. What is the highest rate of markdown at which the store will still break-even?
4. What is the highest rate of discount that can be advertised without incurring an absolute loss?

**Solution:** Before we begin the problem let us summarize the given information. From the initial statement we can conclude that  $S = 22.50$ ,  $E = 0.5C$ , and  $P = 0.3C$ .

First we need to find the cost of the concentrate given the above information. We can easily solve this problem by using Equation 8.7.

$$\begin{aligned} S &= C + E + P22.50 && = C + 0.5C + 0.3C \\ 22.50 &= 1.8C \\ \frac{22.50}{1.8} &= C \\ 12.50 &= C \end{aligned}$$

Thus the cost of the concentrate is \$12.50.

Secondly, we need to find the price needed to cover total cost. We can solve this problem by using Equation 8.13.

$$\begin{aligned}T &= C + E \\&= C + 0.5C \\&= 1.5C \\&= 1.50(12.50) \\&= 18.75\end{aligned}$$

Thus the price need to cover total cost (cost and expenses) is \$18.75.

Next, we need to find the highest rate of markdown so that the store will break-even. We will use our previous work to help here. Note again that the regular sell price is \$22.50 and the total cost is \$18.75. Thus the maximum possible markdown to break-even would be the difference of these two values. So  $22.50 - 18.75 = 3.75$ . The question is asking for a rate so we need to convert to markdown rate using Equation 8.15.

$$\begin{aligned}\text{Markdown Rate} &= \frac{MD}{S} \\&= \frac{3.75}{22.50} \\&= 0.1\bar{6}\end{aligned}$$

So the maximum markdown rate to break-even would be  $16.\bar{6}\%$ .

Finally, we need to find the highest rate of discount possible without incurring an absolute loss. Remember, that an absolute loss occurs when the sell price is less than the cost, so we need to find the markdown rate where the sell price equals the cost with respect to the original sell price,  $S$ . The markdown in this case would be  $22.50 - 12.50 = 10$ . So

$$\begin{aligned}\text{Markdown Rate} &= \frac{MD}{S} \\&= \frac{10}{22.50} \\&= 0.\bar{4}\end{aligned}$$

So the maximum rate of markdown for which the company could accept without an absolute loss would be  $44.\bar{4}\%$ . ◀

This concludes our section on Markdown.

## 8.5 Applications-Integrated Problems

This section corresponds to section 5.5 in the textbook. This is the final section within the official course curriculum. Here we are interested in applying all materials from chapter 8 and working problems that could make use of much of what we have discussed in this chapter. Students should have, at the ready, a summary list of formulae from sections 8.1-8.4. A nice list of such formulae can be found in the textbook on pages 222-224.

Since this section is application only, we have no theory to discuss whatsoever. So we will dive right into our problems.

**Example 8.5.1.** [1, p. 209] Rocky Sports purchased ski bindings for \$57.75 that were marked up 45% of the regular selling price. The store's overhead expenses were 28% of the regular selling price. When the binding was discontinued, it was marked down 40%. What was the sale price of the binding? How much was the operating profit or loss as a result of the sale?

**Solution:** Work through the information step by step as it is given. Let us first find the regular selling price using Equation 8.10.

$$\begin{aligned}S &= C + M \\S &= C + 0.45S \\S &= 57.75 + 0.45S \\0.55S &= 57.75 \\S &= \frac{57.75}{0.55} \\S &= 105\end{aligned}$$

So the regular selling price is \$105.00.

Now that we have the regular selling price we can use it to find the new sell price using Equation 8.14.

$$\begin{aligned}N &= S - MD \\&= S - 0.4S \\&= 105 - 0.4(105) \\&= 63\end{aligned}$$

So the new selling price is \$63 and we have answered the first question in the problem.

Next, we need to find the amount of operating profit or loss based on our work above and information in the problem. We can use Equation 8.7.

$$\begin{aligned}S &= C + E + P \\63 &= 57.75 + 0.28(105) + P \\-24.15 &= P\end{aligned}$$

Thus, with the sale price, there will be an operating loss of \$24.15. ◀

**Example 8.5.2.** Anna has a bookstore and is working on setting a price for a particular fast selling hardback book. She purchases the books for her store from a wholesaler for \$20 less 10% and 3%. Expenses are on average 30% of the regular selling price and net profit is 25% of the regular selling price. She has the book marked for sale at with a discount (markdown) of 15%. What is the sale price? What is the total cost? What is the profit, if any?

**Solution:** Again, work through the problem in pieces. We begin with finding the cost to Anna's bookstore. We will use the NPF method. So

$$\begin{aligned} C &= 20(0.90)(0.97) \\ &= 17.46 \end{aligned}$$

So the cost,  $C$ , is \$17.46.

What else do we know? We know  $E = 0.30S$  and  $P = 0.25S$ . From this we we can find  $S$  using Equation 8.7.

$$\begin{aligned} S &= C + E + P \\ S &= 17.46 + 0.3S + 0.25S \\ 0.45S &= 17.46 \\ S &= \frac{17.46}{0.45} \\ S &= 38.8 \end{aligned}$$

So the sale price is \$38.80.

Now to find the total cost we use Equation 8.13.

$$\begin{aligned} T &= C + E \\ &= 17.46 + 0.3S \\ &= 17.46 + 0.3(38.8) \\ &= 29.1 \end{aligned}$$

So the total cost is \$29.10.

Finally, to find the profit or loss we can use Equation 8.7 in conjunction with Equation 8.13.

$$\begin{aligned} S &= C + E + P \\ S &= T + P && \text{Substitute using Equation 8.13} \\ 38.8 &= 29.1 + P \\ 9.7 &= P \end{aligned}$$

So the operating profit will be \$9.70 per book. ◀

**Example 8.5.3.** [1, p. 215] A jewelry store paid \$36.40 for a watch. Store expenses are 24% of regular selling price and the normal net profit is 20% of regular selling price. During a Special Bargain Day Sale, the watch was sold at a discount of 30%. What operating profit or loss was realized on the sale?

**Solution:** We should begin this problem by using Equation 8.7 to find the sale price.

$$\begin{aligned}S &= C + E + P \\S &= C + 0.24S + 0.2S \\S &= 36.4 + 0.24S + 0.2S \\0.56S &= 36.4 \\S &= \frac{36.4}{0.56} \\S &= 65\end{aligned}$$

So the regular sale price is \$65.

To find the operating profit/loss we first need to know what the watch was sold for. To find this we will use Equation 8.14.

$$\begin{aligned}N &= S - MD \\&= 65 - 0.3(65) \\&= 45.5\end{aligned}$$

So the new sale price is \$45.50. With the new sale price we can find the profit/loss by subtracting the total cost from this new sale price. Thus

$$\begin{aligned}P &= N - T \\&= N - (C + E) \\&= N - C - E \\&= N - 36.40 - 0.24S \\&= 45.50 - 36.40 - .24(65) \\&= -6.5\end{aligned}$$

Thus we have an operating loss of \$6.50. Concluding our problem. ◀

This also concludes the chapter and brings us to the end of the required course material.

## Bibliography

- [1] S. A. Hummelbrunner, K. Suzanne Coombs, Kelly Halliday, Contemporary Business Mathematics with Canadian Applications, Pearson Canada, Toronto, 9th Edition, 2012.